DESCRIBING FUNCTION METHOD

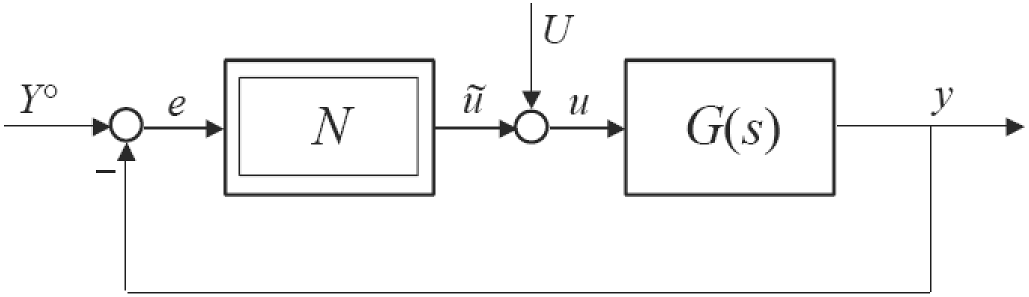
LIMIT CYCLES

Goals:

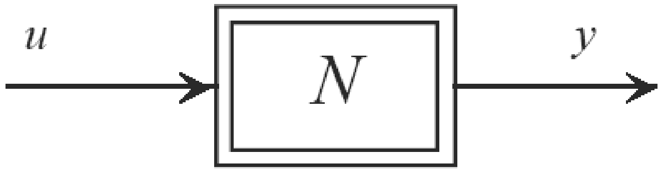
Provide conditions to assess

- existence
- amplitude
- stability

of periodic solutions in a time-invariant Lur'e system subject to constant inputs



DESCRIBING FUNCTION: DEFINITIONS



Sinusoidal-input describing function

- Sinusoidal input $u(t) = U\,cos(\Omega\, t)$
- Periodic solution of nonlinear system *N* $y_p(t;\, U,\, \Omega)$

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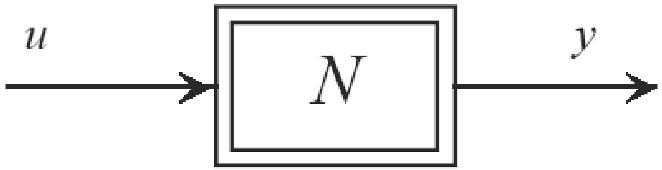

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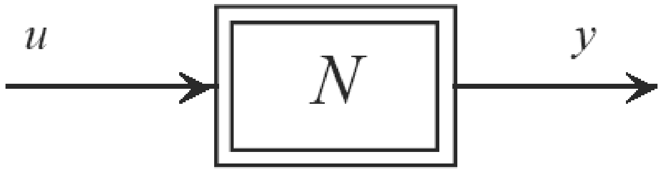
  Remark:
  we assume that it is well-defined and unique for each *U* and Ω

DESCRIBING FUNCTION: DEFINITIONS



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FOURIER SERIES

Periodic function *f(t)* of period T, angular frequency $\Omega = 2\pi/T$

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$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\Omega t) + b_k \sin(k\Omega t)),$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(k\Omega t) dt, \;\; k = 0, 1, 2, \ldots$$

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$$A_k := \sqrt{a_k^2 + b_k^2}$$

$$a_k \cos(k\Omega t) + b_k \sin(k\Omega t) = A_k \left(\frac{a_k}{A_k} \cos(k\Omega t) + \frac{b_k}{A_k} \sin(k\Omega t) \right)$$

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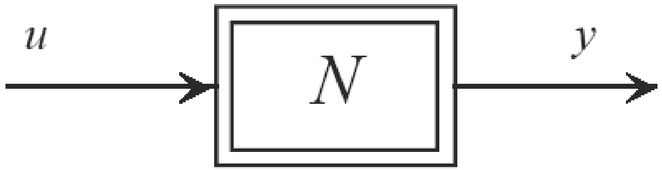
$$\cos(\phi_k) = \frac{a_k}{A_k},\quad \sin(\phi_k) = -\frac{b_k}{A_k}$$

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\Omega t + \phi_k)$$

DESCRIBING FUNCTION: DEFINITIONS



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Fourier series of function $y_p(t;\, U,\, \Omega)$

$$y_p(t;\, U,\, \Omega) = Y_0(U,\Omega) + Y_1(U,\Omega)\, cos\big(\Omega\, t + \varphi_1(U,\Omega)\big) + \\ + \sum_{k=2}^{\infty} Y_k(U,\Omega)\, cos\big(k\,\Omega\, t + \varphi_k(U,\Omega)\big)$$

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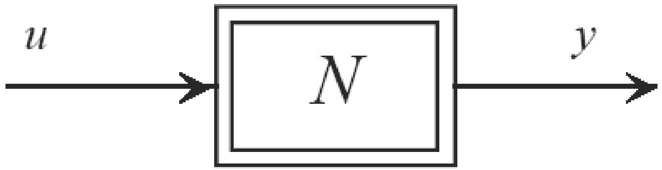

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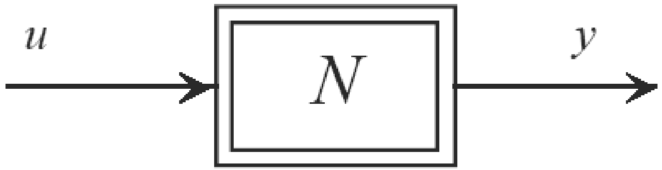
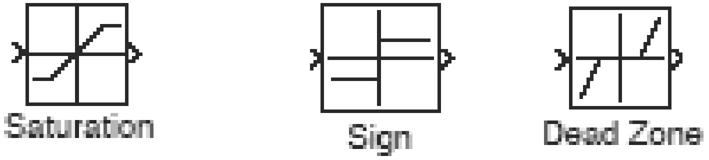
$$D(U,\, \Omega) := \frac{Y_1(U,\Omega)}{U}\, e^{j\varphi_1(U,\Omega)}$$

DESCRIBING FUNCTION: DEFINITIONS



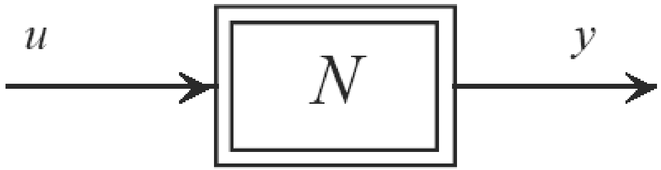
We consider nonlinear systems that are described by some input-output characteristic function

Memoryless nonlinearity:



Nonlinearity with memory:

DESCRIBING FUNCTION: DEFINITIONS



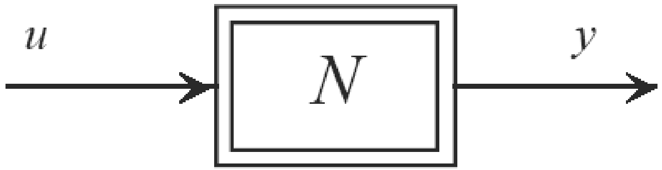
We consider nonlinear systems that are described by some input-output characteristic function

Properties:

- The describing function of *N* is independent of Ω
- If the input-output function *N* is a single value function ($y = f(u)$), then, the describing function takes values in R

$$D(U,\Omega) \rightarrow D(U)$$

DESCRIBING FUNCTION: DEFINITIONS



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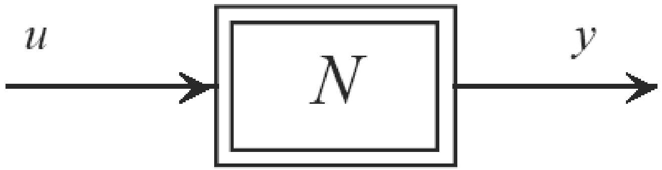

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$$y_p(t;\, U_0,\, U_1,\, \Omega) = Y_0(U_0,\, U_1,\, \Omega) +$$

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$$+ \sum_{k=2}^{\infty} Y_k(U_0,\, U_1,\, \Omega)\, cos\big(k\, \Omega\, t + \varphi_k(U_0,\, U_1,\, \Omega)\big)$$

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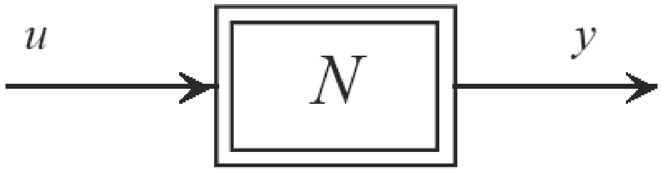

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- Input $u(t) = U_0 + U_1\, cos(\Omega\, t)$
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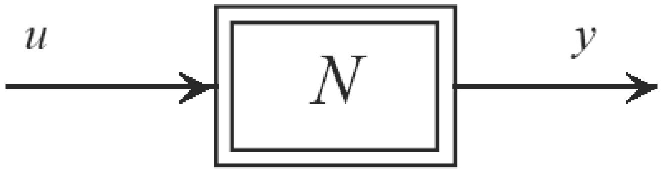
$$D_0(U_0, U_1, \Omega) := \frac{Y_0(U_0, U_1, \Omega)}{U_0}$$

$$D_1(U_0, U_1, \Omega) := \frac{Y_1(U_0, U_1, \Omega)}{U_1}\; e^{j\varphi_1(U_0, U_1, \Omega)}$$

DESCRIBING FUNCTION: DEFINITIONS



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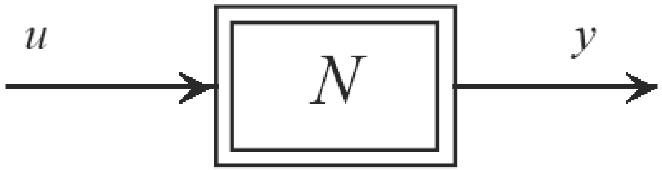
Properties:

- The describing functions of *N* are independent of Ω

$$D_0(U_0, U_1, \Omega) \to D_0(U_0, U_1) \qquad D_1(U_0, U_1, \Omega) \to D_1(U_0, U_1)$$

- If the input-output function *N* is a single value function ($y = f(u)$), then, both describing functions take values in R

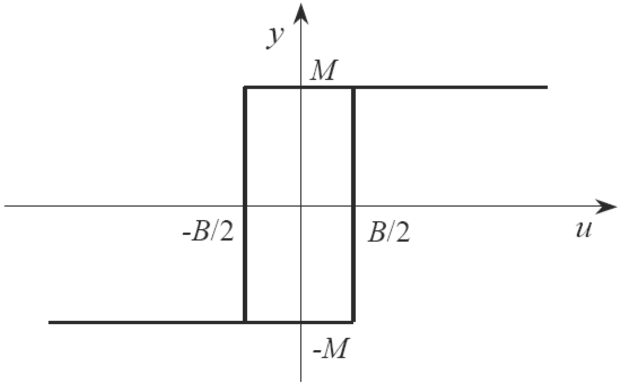
DESCRIBING FUNCTION: DEFINITIONS



Proposition

- The describing functions of two nonlinearities in parallel are given by the sum of the describing functions of the two nonlinearities

EXAMPLE: TWO-LEVEL RELAY WITH HYSTERESIS MB/2



- Sinusoidal-input describing function

$$D(U) = \frac{2\,M}{\pi\,U^2}\left(\sqrt{4\,U^2 - B^2} - j\,B\right) \qquad U \geq B/2$$

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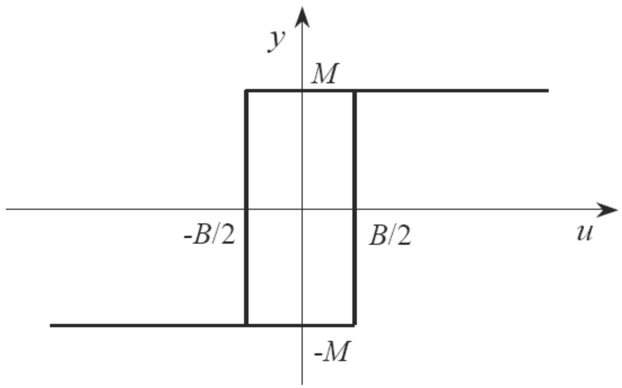

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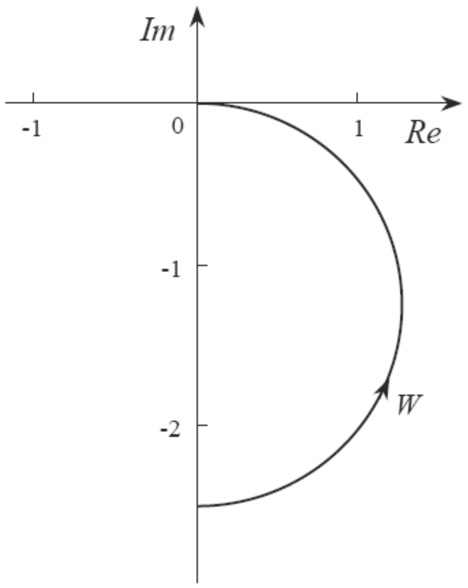
$$D(U) = \frac{M}{B} D^*(W) \quad , \quad W := \frac{U}{B} \geq 1/2$$

$$D^*(W) := \frac{2}{\pi} \frac{\sqrt{4W^2 - 1} - j}{W^2}$$



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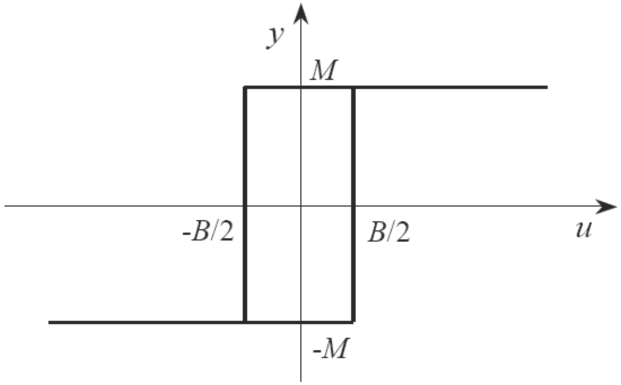

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If $B = 0$,

$$D(U) = \frac{4M}{\pi U} \quad , \qquad U > 0$$

EXAMPLE: TWO-LEVEL RELAY WITH HYSTERESIS MB/2

- Dual-input describing functions $u(t) = U_0 + U_1\, cos(\Omega\, t)$

$$D_0(U_0, U_1) = \frac{M}{U_0}\left(g_0(\sigma) - g_0(\delta)\right)$$

$$D_1(U_0, U_1) = \frac{2\,M}{\pi\, U_1}\left(g_1(\sigma) + g_1(\delta) - j\,\frac{B}{U_1}\right)$$

where

$$g_0(x) := \begin{cases} -1/2 & , \quad x < 1 \\ 1/\pi\ arcsin(x) & , \quad |x| \le 1 \\ 1/2 & , \quad x > 1 \end{cases} \qquad g_1(x) := \begin{cases} \sqrt{1 - x^2} & , \quad |x| \le 1 \\ 0 & , \quad |x| > 1 \end{cases}$$

$$\sigma := a + r \qquad \delta := a - r \qquad a := \frac{B}{2\,U_1} \qquad r := \frac{U_0}{U_1}$$

$$U_1 - |U_0| \ge \frac{B}{2}$$

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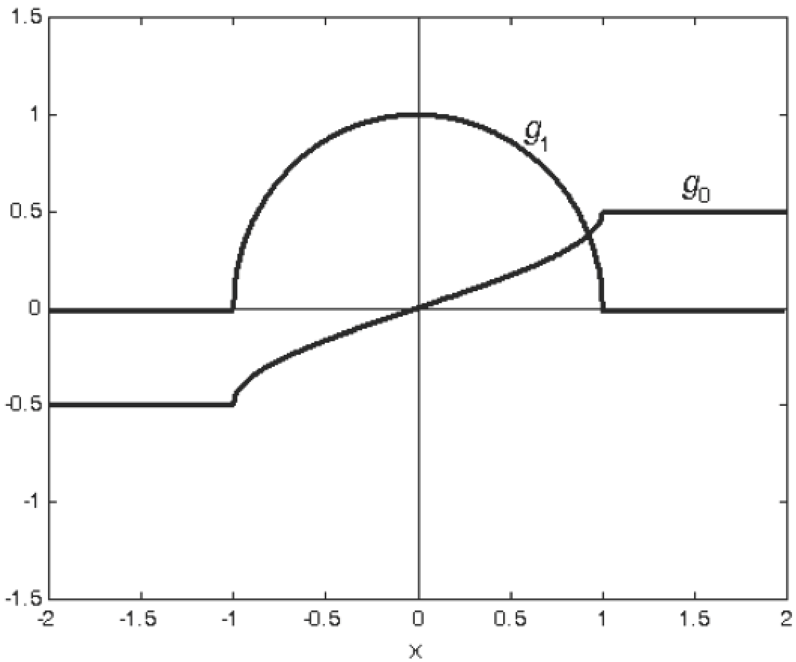

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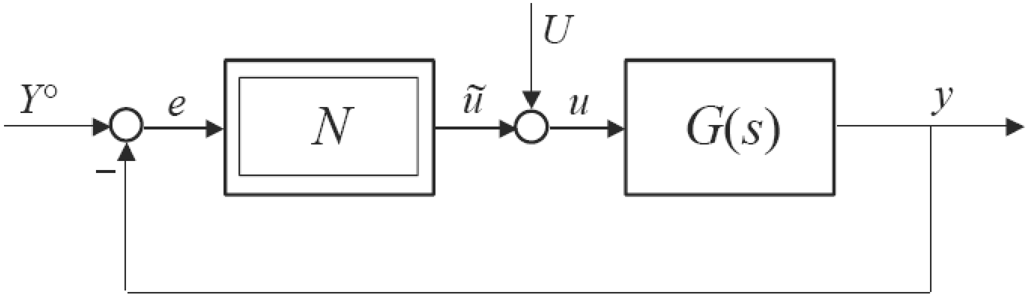
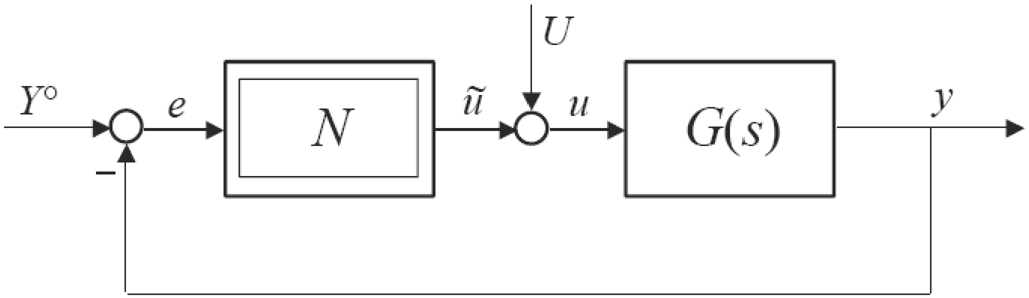
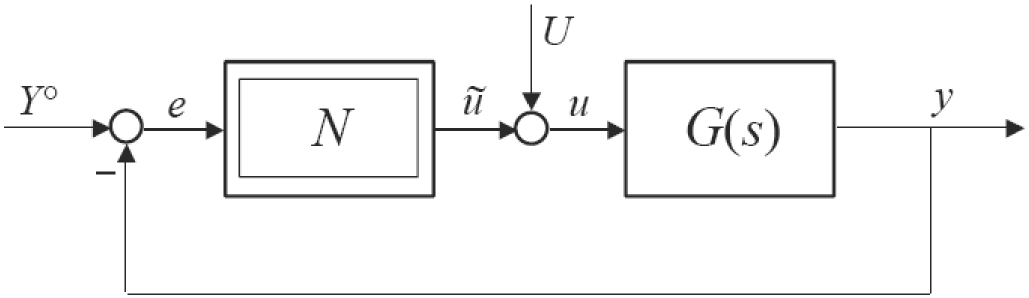


DESCRIBING FUNCTION METHOD



Does there exist a periodic solution associated with costant inputs U and Y°?

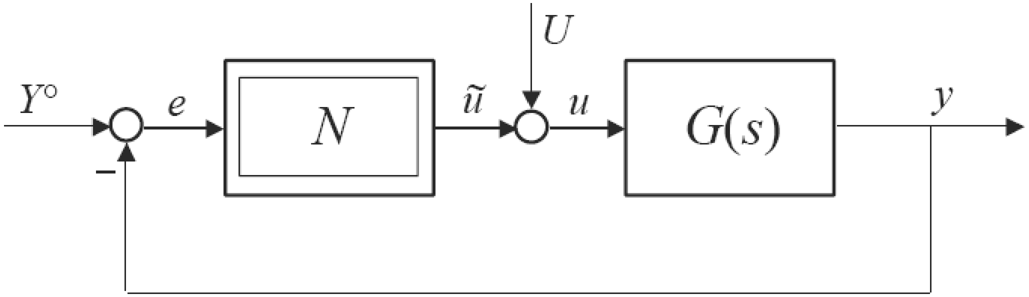
DESCRIBING FUNCTION METHOD



Does there exist a periodic solution associated with constant inputs U and Y°?

Let us assume that there does exist and that it has period T

PERIODIC SOLUTIONS IN A LUR'E SYSTEM



If there exists a periodic solution with period T, then

$$u(t) = U_0 + U_1\, cos(\Omega\, t + \beta_1) + \sum_{k=2}^{\infty} U_k\, cos(k\,\Omega\, t + \beta_k) \qquad \Omega := 2\,\pi/T;$$

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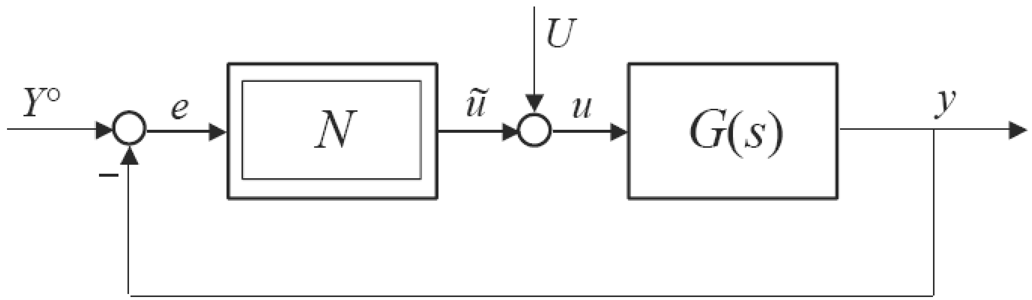

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Correspondingly, we have

$$y(t) = G_0 \, U_0 + G_1 \, U_1 \, cos(\Omega \, t + \beta_1 + \gamma_1) + \sum_{k=2}^{\infty} G_k \, U_k \, cos(k \, \Omega \, t + \beta_k + \gamma_k)$$

where $G_k := |G(jk\Omega)|$ and $\gamma_k := \angle G(jk\Omega)$

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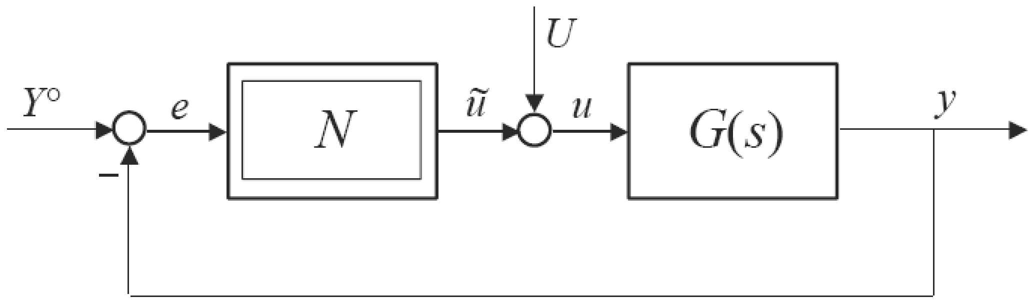

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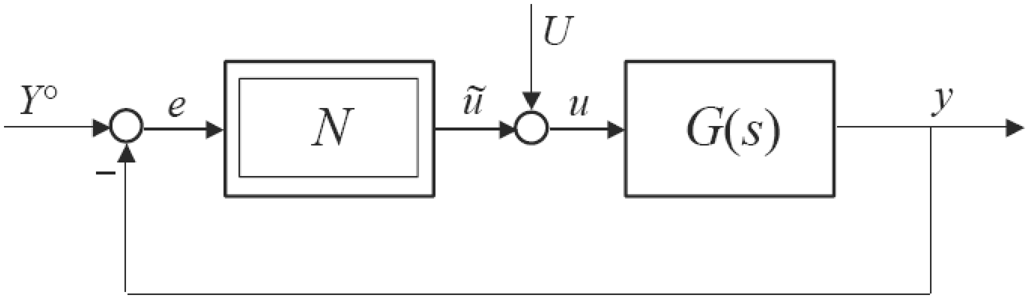
Filtering assumption:

Assume that $G_k\, U_k << G_1\, U_1 \quad , \quad \forall\, k \geq 2$

PERIODIC SOLUTIONS IN A LUR'E SYSTEM



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Under the filtering assumption, we get

$$y(t) \approx G_0\, U_0 + G_1\, U_1\, cos(\Omega\, t + \beta_1 + \gamma_1)$$

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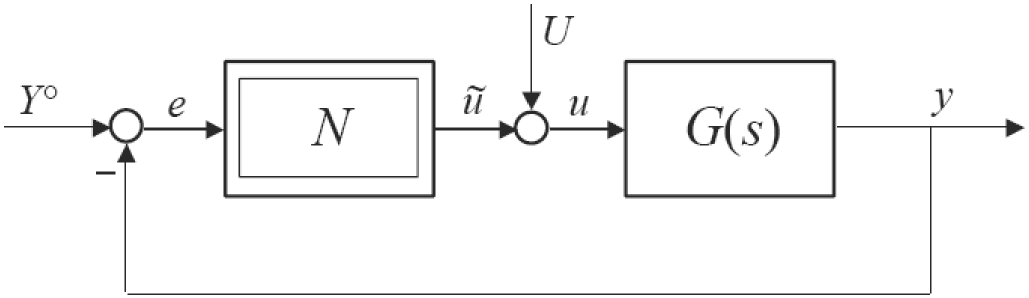

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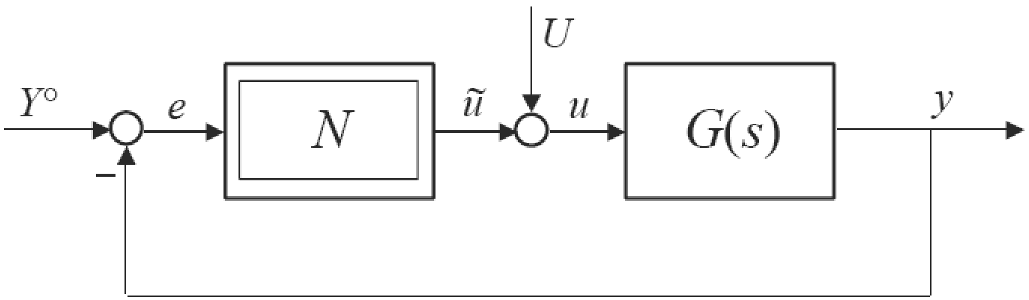
$$y(t) \approx G_0\, U_0 + G_1\, U_1\, cos(\Omega\, t + \beta_1 + \gamma_1)$$

By suitably setting the time origin

$$e(t) \approx Y^\circ - G_0\, U_0 - G_1\, U_1\, cos(\Omega\, t + \beta_1 + \gamma_1) \rightarrow E_0 + E_1\, cos(\Omega\, t)$$

PERIODIC SOLUTIONS IN A LUR'E SYSTEM



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By suitably setting the time origin

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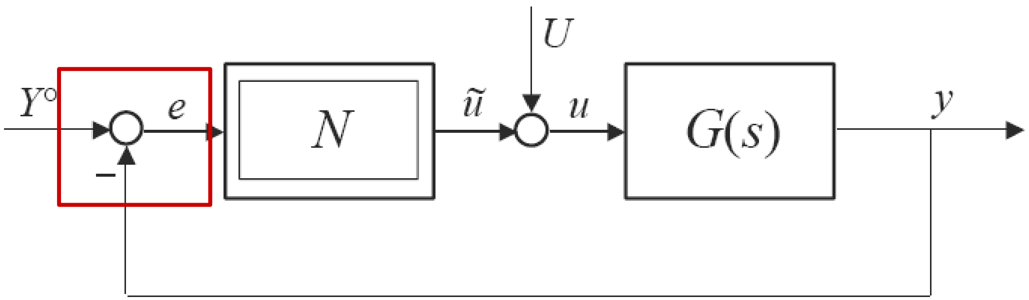
→ the input to *N* is the sum of a constant and a fundamental harmonic contribution

→ need only the mean and first harmonic signal of the output of *N*

HARMONIC BALANCE EQUATIONS

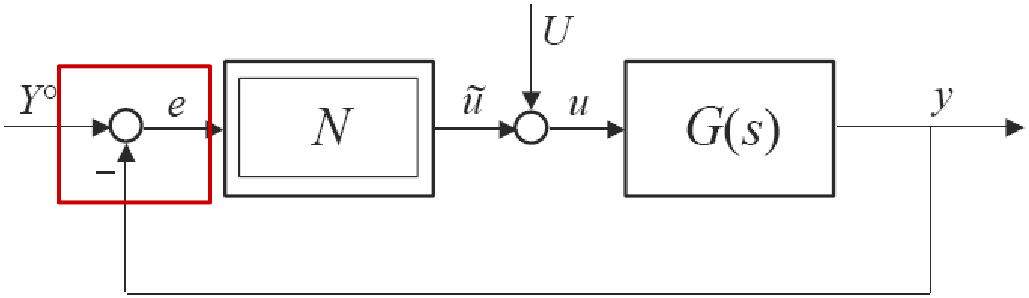


1) $(g \leq 0)$ $\quad E_0 = Y° - G_0\,(U + D_0\,(E_0, E_1, \Omega)\,E_0)$

balance of the average value

average of the output of N

HARMONIC BALANCE EQUATIONS



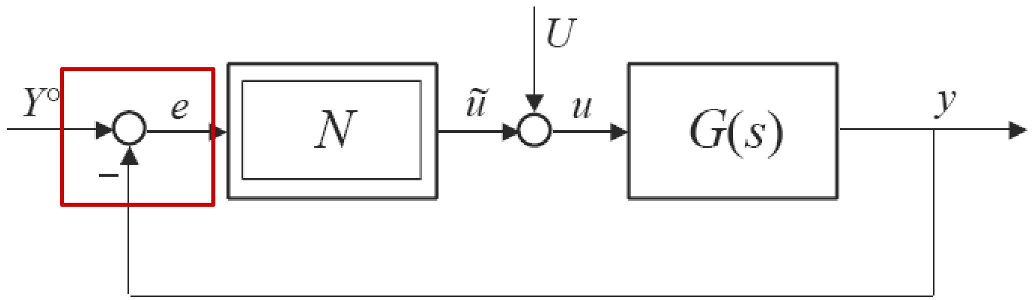
1) $(g \leq 0)$ $$E_0 = Y° - G_0\,(U + D_0\,(E_0, E_1, \Omega)\,E_0)$$

balance of the average value

If G(s) has zero poles, then $G_0 \rightarrow \infty$ and the balance of the average value equation becomes:

1') $(g > 0)$ $$U + D_0\,(E_0, E_1, \Omega)\,E_0 = \frac{Y° - E_0}{G_0} = 0$$

HARMONIC BALANCE EQUATIONS



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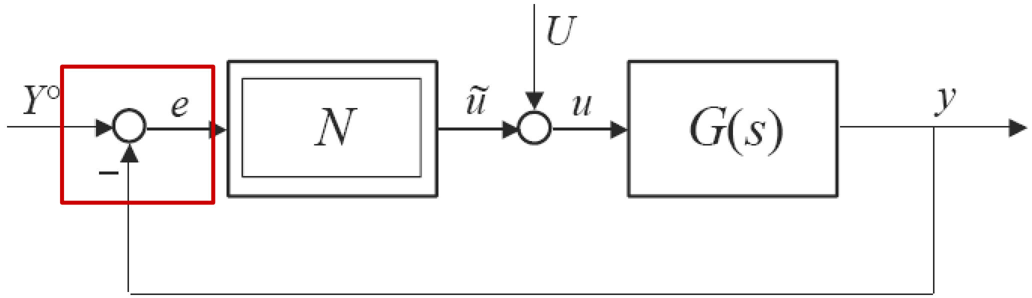
HARMONIC BALANCE EQUATIONS



1) $$U + D_0\,(E_0, E_1, \Omega)\,E_0 = \frac{Y^\circ - E_0}{G_0}$$

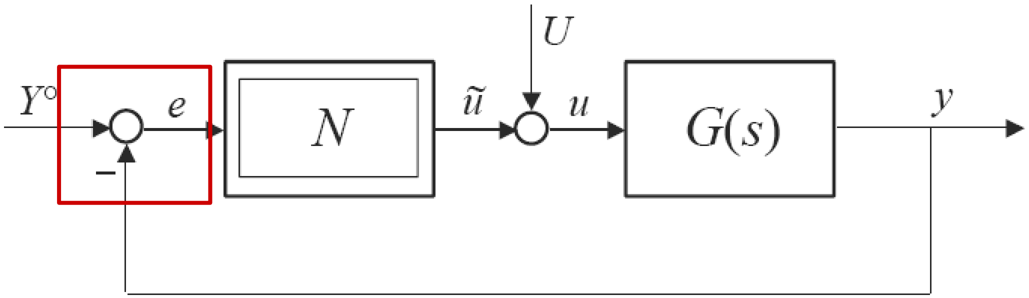
balance of the average value

2) $$E_1 = -\,G(j\Omega)\,D_1(E_0, E_1, \Omega)\,E_1$$

balance of the first harmonic

polar representation of the first harmonic

HARMONIC BALANCE EQUATIONS



Balance of the average value $$U + D_0\,(E_0, E_1, \Omega)\,E_0 = \frac{Y^{\circ} - E_0}{G_0}$$

Balance of the first harmonic $$E_1 = -\,G(j\Omega)\,D_1(E_0, E_1, \Omega)\,E_1$$

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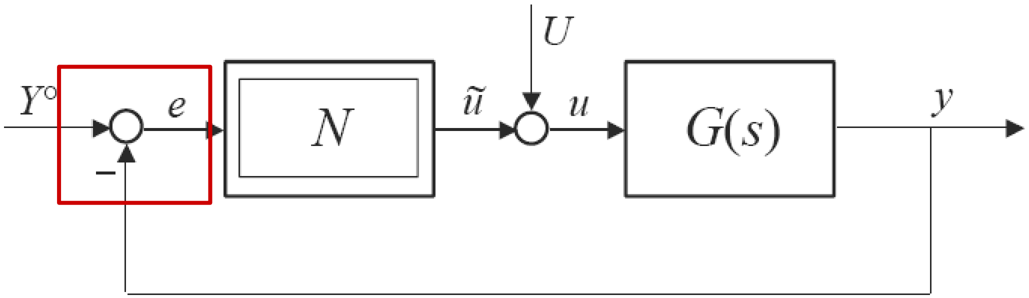

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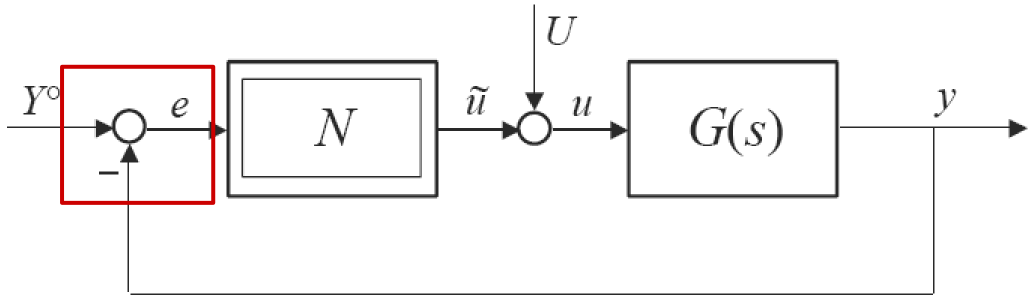

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→ 3 equations between real numbers in E_0, E_1, Ω

<u>Remarks:</u>

- If we know *e(t),* we can then determine all signals
- Nonlinear algebraic equations
  → no simple conditions for existence and uniqueness of the solution, neither analytical formulas. Typically, numerical solutions are adopted

EXAMPLE: *N* DESCRIBED BY INPUT-OUTPUT MAP

$$U + D_0\,(E_0, E_1)\,E_0 = \frac{Y^\circ - E_0}{G_0}$$

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(1) o (1') $\varphi(E_0, E_1) = 0 \quad \Leftrightarrow \quad E_0 = \eta(E_1)$

(2) $(E_1 \neq 0)$ $E_1 = -\,G(j\Omega)\,D_1(E_0, E_1)\,E_1$

$$G(j\Omega) = \frac{-1}{D_1(E_0, E_1)} = \frac{-1}{D_1(\eta(E_1), E_1)} := H_1(E_1)$$

EXAMPLE: *N* DESCRIBED BY INPUT-OUTPUT MAP

$$U + D_0\,(E_0, E_1)\,E_0 = \frac{Y^\circ - E_0}{G_0}$$

$$G(j\Omega)\,D_1(E_0, E_1) = -1\ .$$

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$G(j\Omega)$ ↓ polar plot Γ

$H_1(E_1)$ ↓ DF plot

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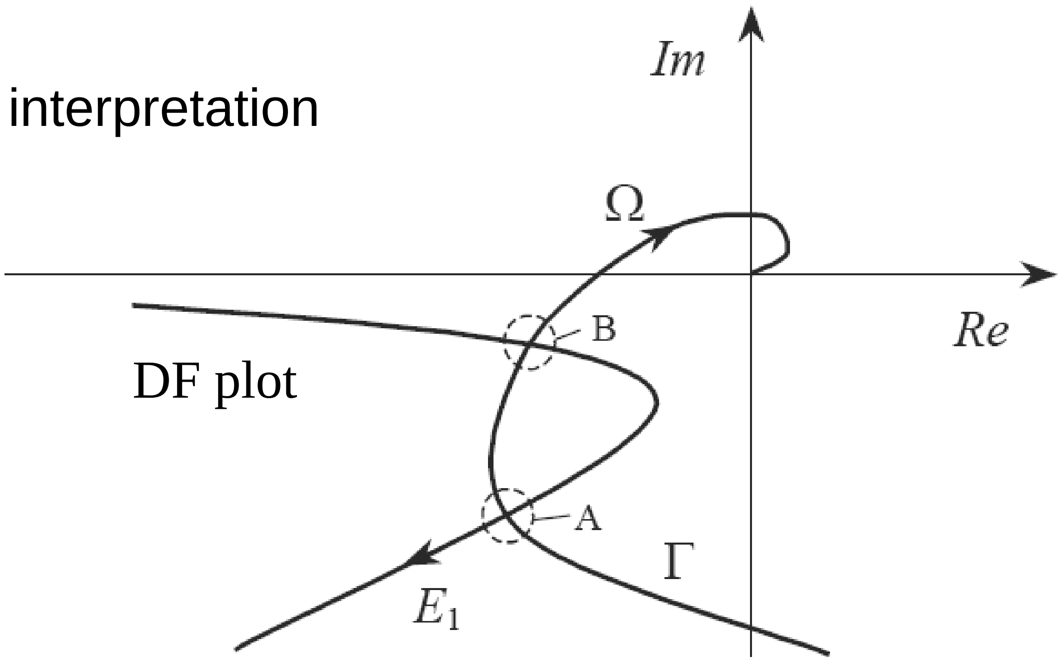
Graphical interpretation



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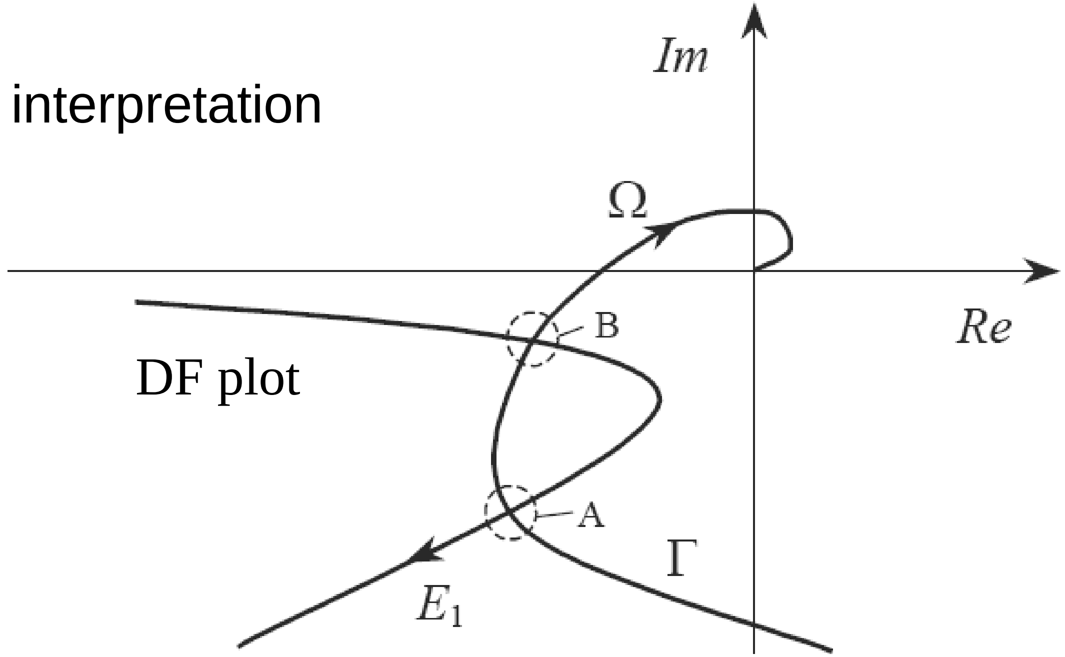

- Ω_A, E_{1A}, $\quad E_{0A} = \eta(E_{1A}) \rightarrow e_A(t) = E_{0A} + E_{1A} \cos(\Omega_A t)$
- Ω_B, E_{1B}, $\quad E_{0B} = \eta(E_{1B}) \rightarrow e_B(t) = E_{0B} + E_{1B} \cos(\Omega_B t)$

EXAMPLE: *N* DESCRIBED BY INPUT-OUTPUT MAP

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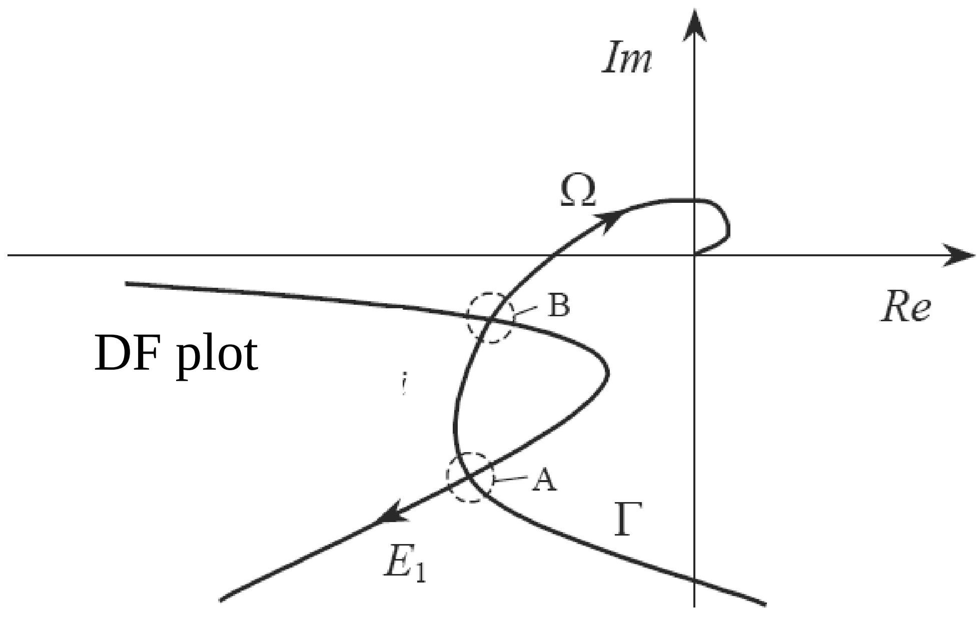
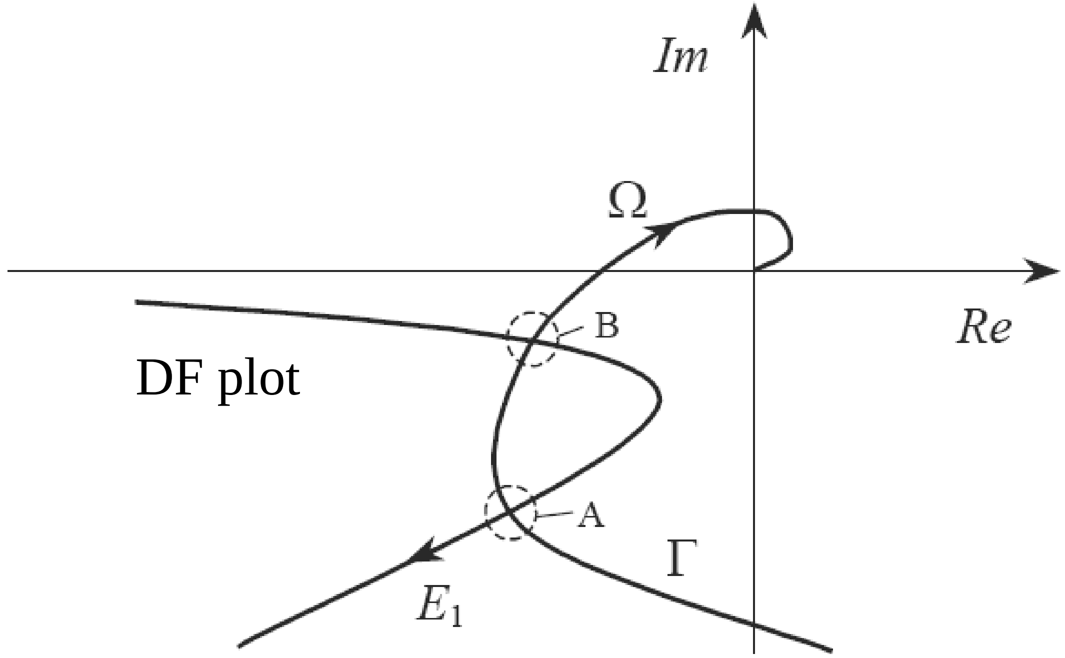


EXAMPLE: *N* DESCRIBED BY INPUT-OUTPUT MAP

Remark [robustness]:

If the two plots intersect, then, they will keep intersecting even in presence of small perturbations of the two systems



→ Robustness of the limit cycle, in contrast with the linear systems case

PARTICULAR CASE: $E_0 << E_1$

$$E_0 \cong 0 \qquad \Rightarrow \qquad e(t) \cong E\, cos(\Omega\, t)$$

Harmonic balance equation

$$E = -\, G(j\Omega)\, D(E,\Omega)\, E \qquad \Leftrightarrow \qquad G(j\Omega) = \frac{-1}{D(E,\Omega)}$$

sinusoidal-input describing function

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sinusoidal-input describing function

$$1 + G(j\Omega)D(E,\Omega) = 0$$

- pseudo-characteristic equation, since it is similar to the characteristic equation for a feedback linear system
- $G(j\Omega)D(E,\Omega)$ plays the role of transfer function of the feedback loop

PARTICULAR CASE: $E_0 << E_1$

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Harmonic balance equation

$$E = -\,G(j\Omega)\,D(E,\Omega)\,E \qquad \Leftrightarrow \qquad G(j\Omega) = \frac{-1}{D(E,\Omega)}$$

If *N* is described by an input-output map, the harmonic balance equation rewrites as

$$G(j\Omega) = \frac{-1}{D(E)} := H(E)$$

$$\downarrow$$

DF plot

PARTICULAR CASE: $E_0 << E_1$

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Harmonic balance equation

$$E = -\, G(j\Omega)\, D(E,\Omega)\, E \qquad \Leftrightarrow \qquad G(j\Omega) = \frac{-1}{D(E,\Omega)}$$

Question: when does the condition $E_0 << E_1$ hold?

PARTICULAR CASE: $E_0 \ll E_1$

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1) $(g \leq 0)$ $\qquad E_0 = Y^\circ - G_0 \left(U + D_0(E_0, E_1, \Omega)\, E_0 \right)$

$$E_0 = \frac{Y^\circ - G_0\, U}{1 + G_0\, D_0(E_0, E_1, \Omega)}$$

1') $(g > 0)$ $\qquad U + D_0(E_0, E_1, \Omega)\, E_0 = 0$

$$E_0 = \frac{-\, U}{D_0(E_0, E_1, \Omega)}$$

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Harmonic balance equation

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choose U so that $E_0 = 0$

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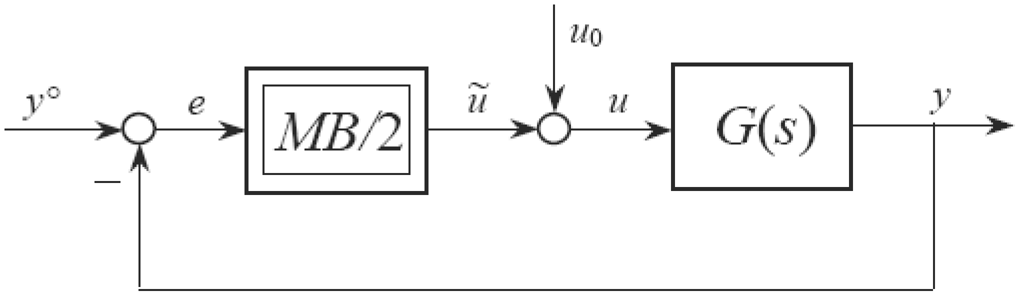
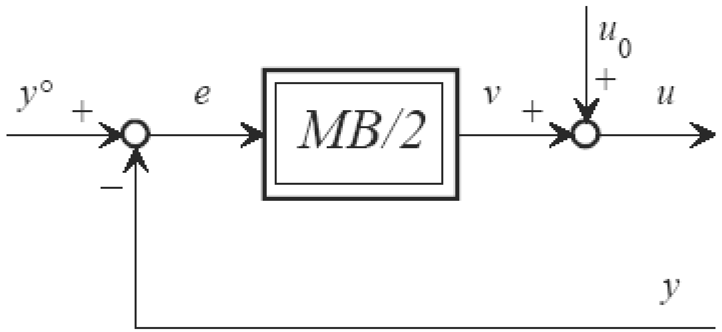
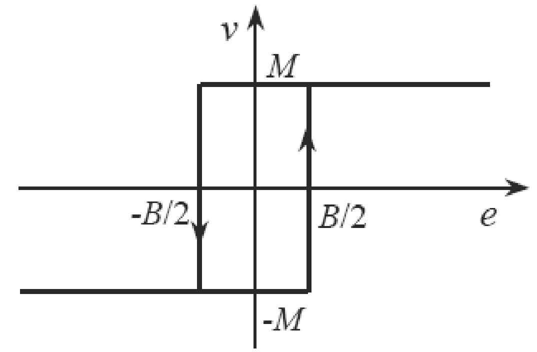


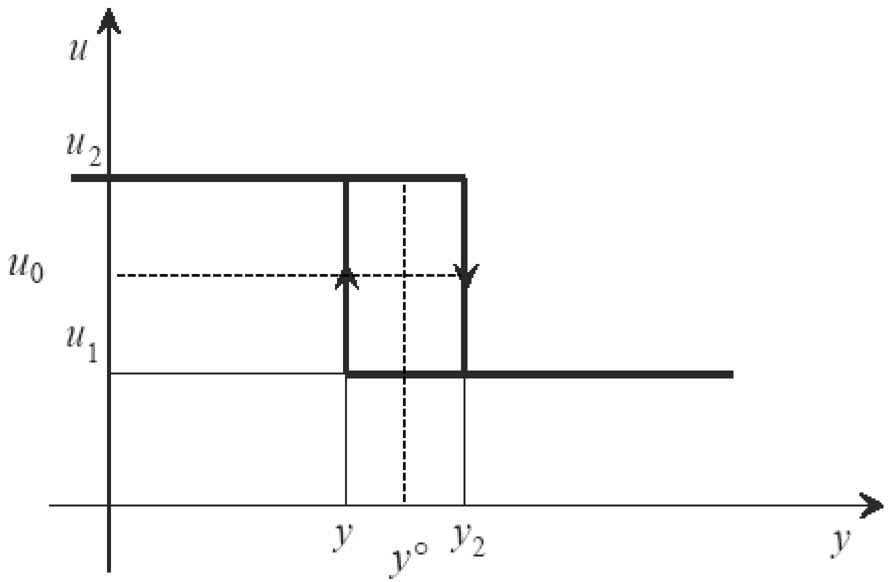
G(s) with no poles equal to zero

Data: y_{min} e y_{max}

TUNING OF THE MB/2 CONTROLLER PARAMETERS

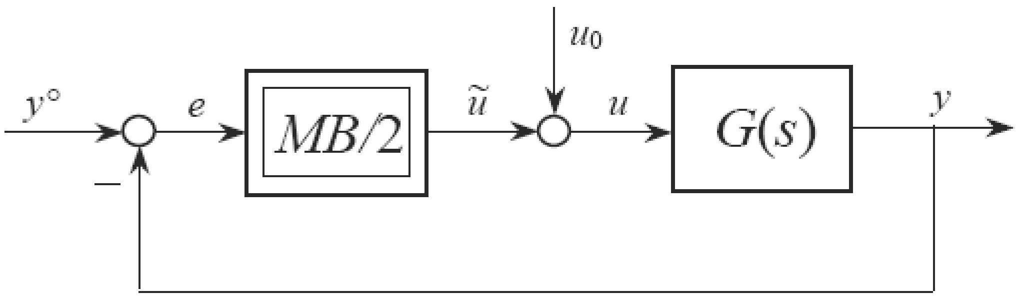






$$u_1 = u_0 - M$$
$$u_2 = u_0 + M$$
$$y_1 = y^\circ - B/2$$
$$y_2 = y^\circ + B/2$$

EXAMPLE



G(s) with no poles equal to zero

Data: y_{min} e y_{max}

Natural choice $y^\circ = (y_{min} + y_{max})/2$

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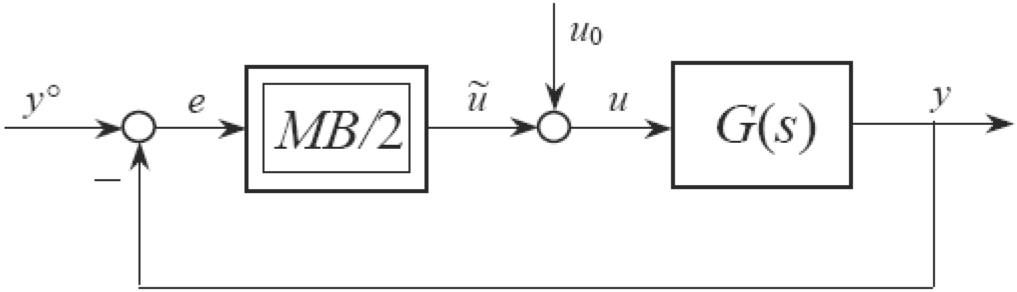

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$$u_0 = \frac{y^\circ}{G_0} \quad \Rightarrow \quad E_0 = 0$$

$$\begin{aligned} u_1 &= u_0 - M \\ u_2 &= u_0 + M \\ y_1 &= y^\circ - B/2 \\ y_2 &= y^\circ + B/2 \end{aligned}$$

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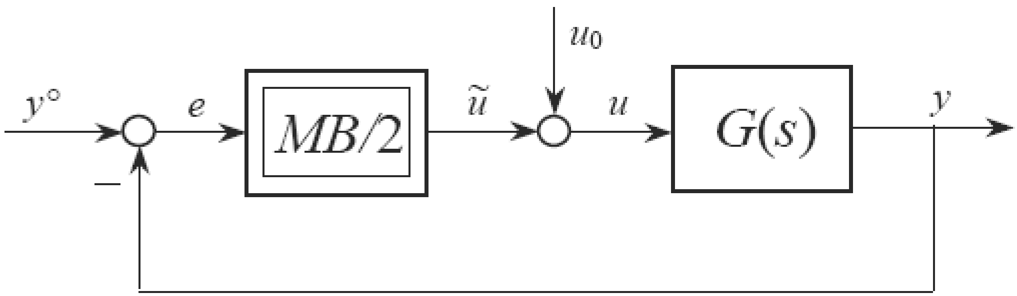

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$$B: \quad y_1 = y^\circ - B/2 \geq y_{min}$$

$$M: \quad u_1 = u_0 - M \quad \rightarrow \quad y_\infty < y_1$$

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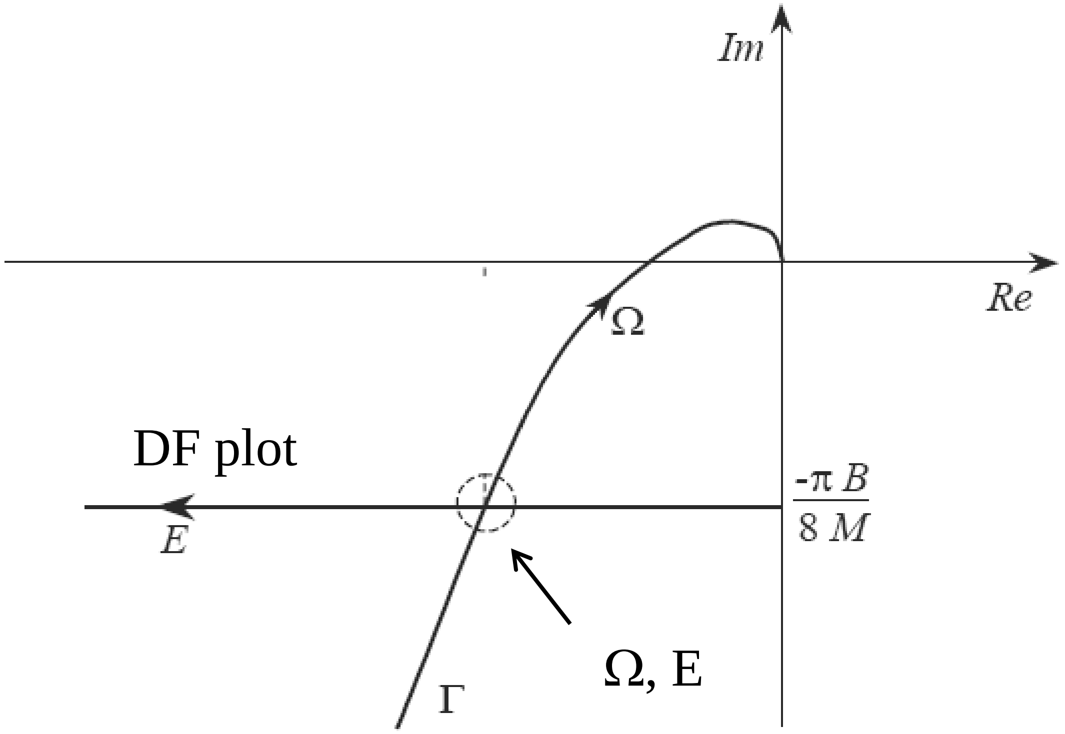

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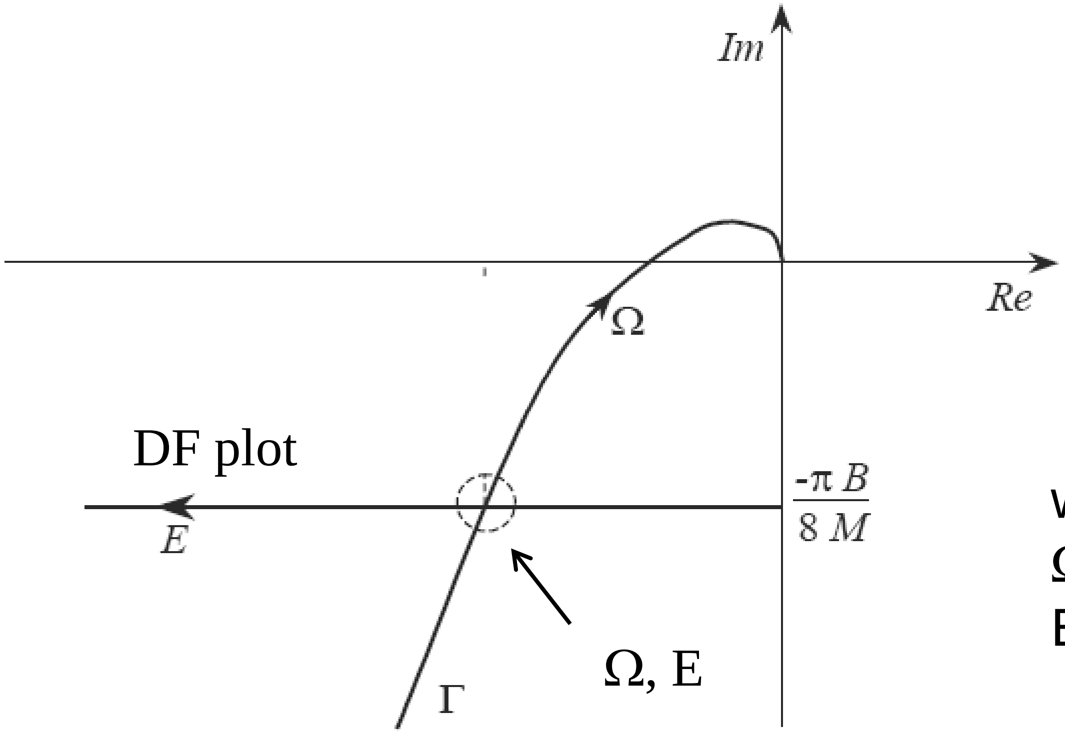
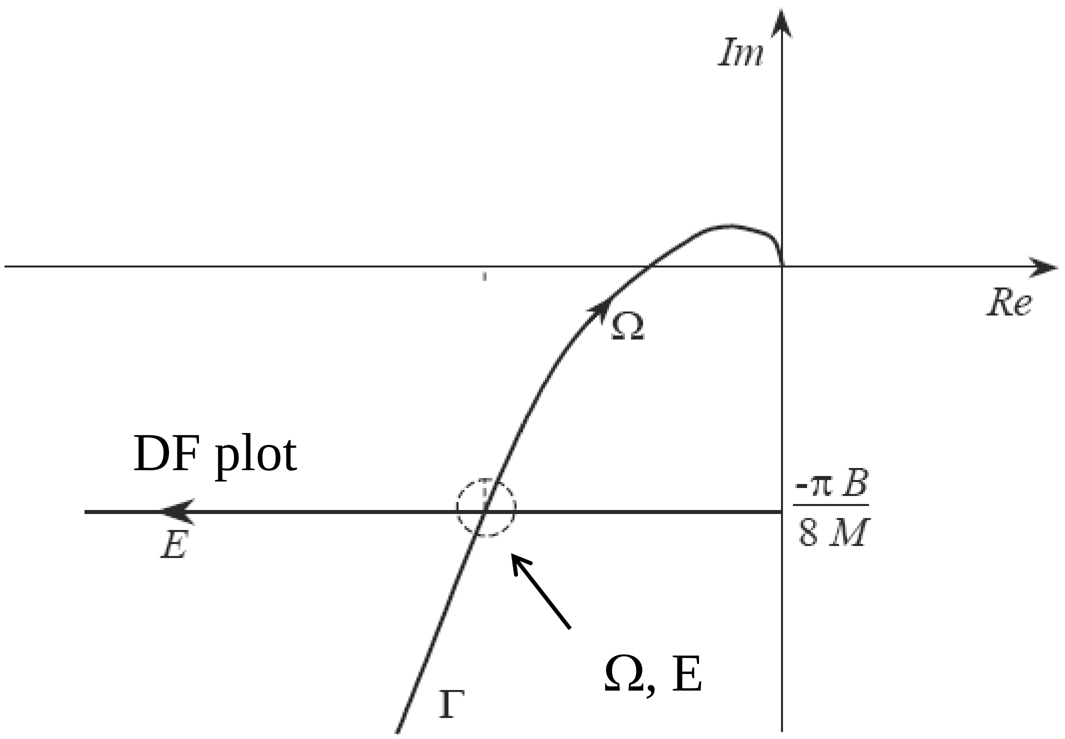


when B/M decreases,
Ω increases and
E decreases

EXAMPLE

Let B and M be fixed. Then Ω is derived as the angular frequency at which the polar plot crosses the horizontal axis crossing the imaginary axis at $\frac{-\pi B}{8M}$



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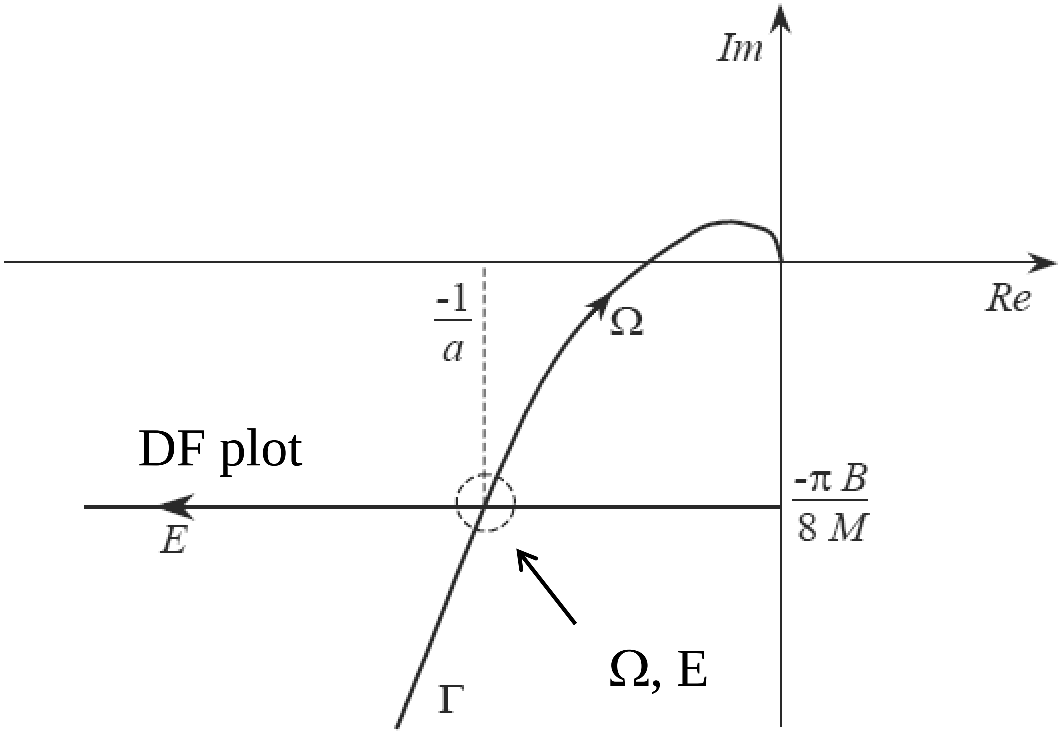

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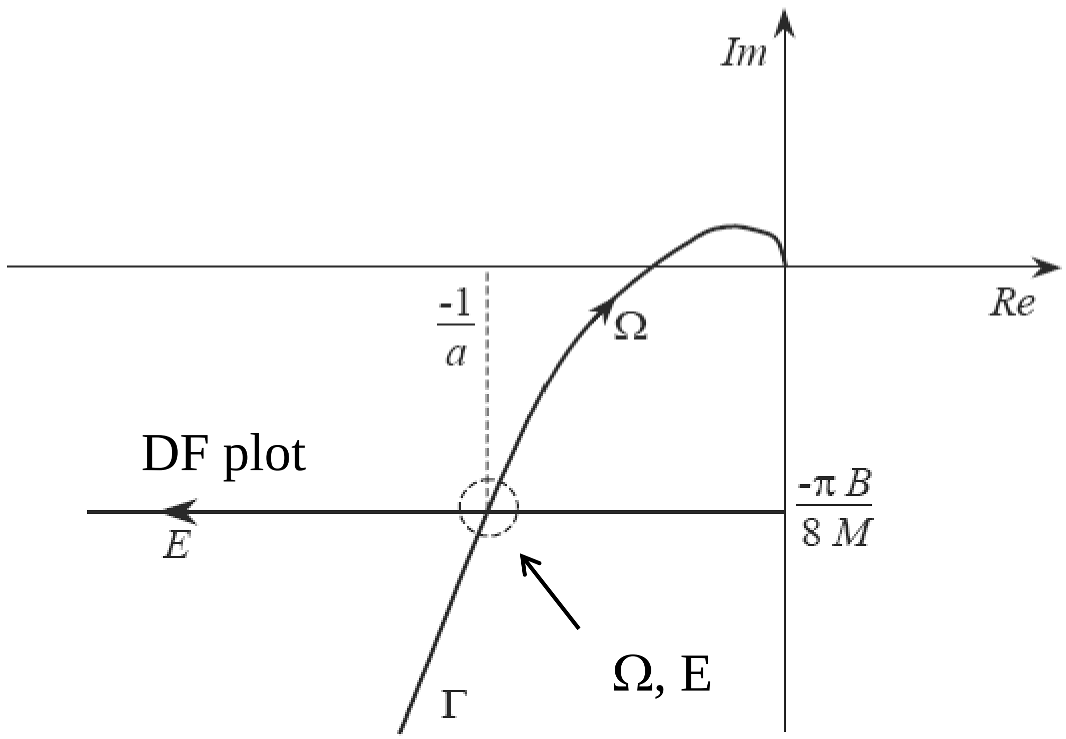
$$H(E) = -\frac{\pi}{8M}(\sqrt{4E^2 - B^2} + jB) \rightarrow -\frac{\pi}{8M}\sqrt{4E^2 - B^2} = -\frac{1}{a}$$



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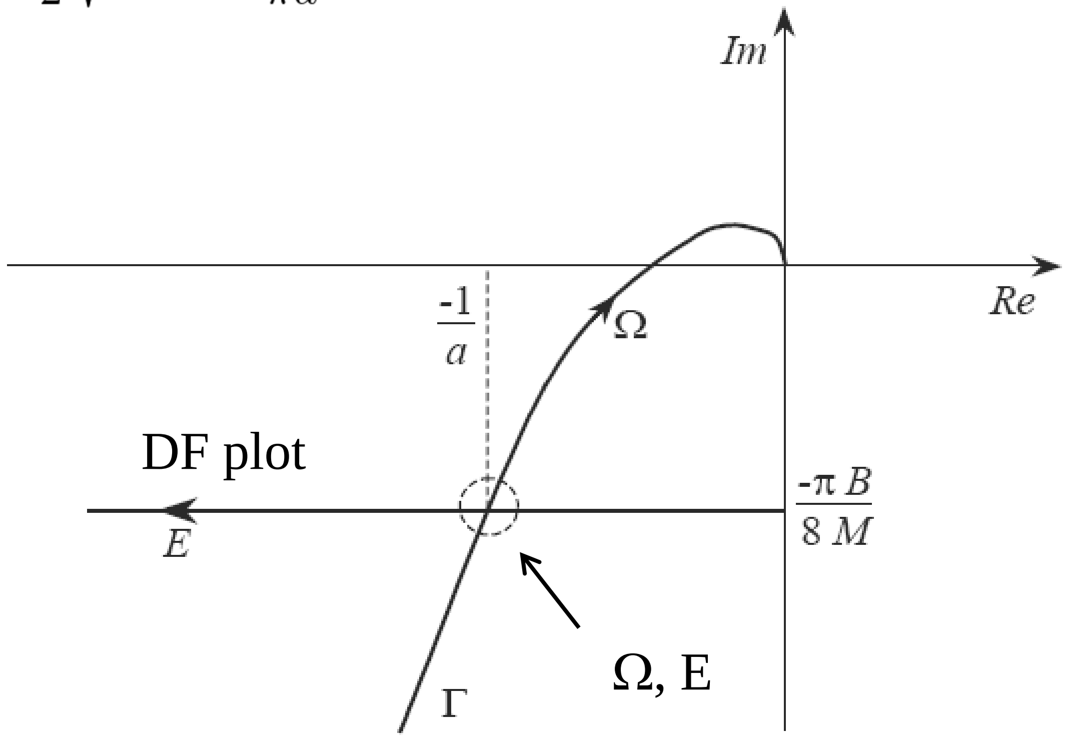
$$\Rightarrow E = \frac{1}{2}\sqrt{B^2 + (\frac{8M}{\pi a})^2}$$

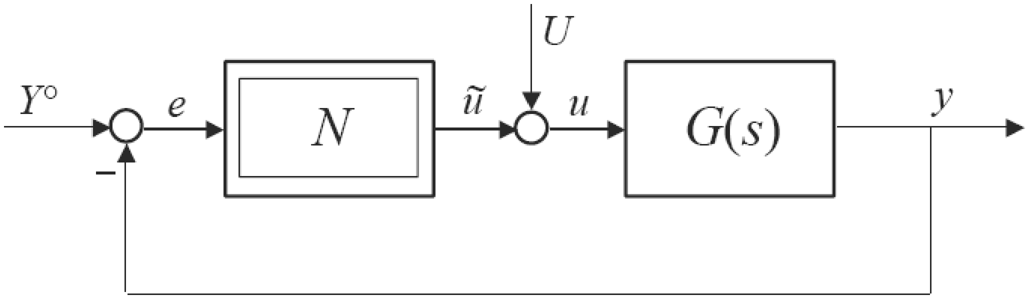


EXAMPLE

Remark:

Heuristic approach, based on the filtering assumption, that depends in turn on the solution to the problem…

PERIODIC SOLUTIONS IN A LUR'E SYSTEM



If there exists a periodic solution with period T, then

$$u(t) = U_0 + U_1\, cos(\Omega\, t + \beta_1) + \sum_{k=2}^{\infty} U_k\, cos(k\, \Omega\, t + \beta_k) \qquad \Omega := 2\, \pi / T$$

Correspondingly, we have

$$y(t) = G_0\, U_0 + G_1\, U_1\, cos(\Omega\, t + \beta_1 + \gamma_1) + \sum_{k=2}^{\infty} G_k\, U_k\, cos(k\, \Omega\, t + \beta_k + \gamma_k)$$

where $G_k := |G(jk\Omega)|$ and $\gamma_k := \angle G(jk\Omega)$

Filtering assumption:

Assume that $G_k\, U_k << G_1\, U_1 \quad , \quad \forall\, k \geq 2$

Remark:

Heuristic approach, based on the filtering assumption, that depends in turn on the solution to the problem…

→ a-posterior analytic assessment

→ validation via simulation

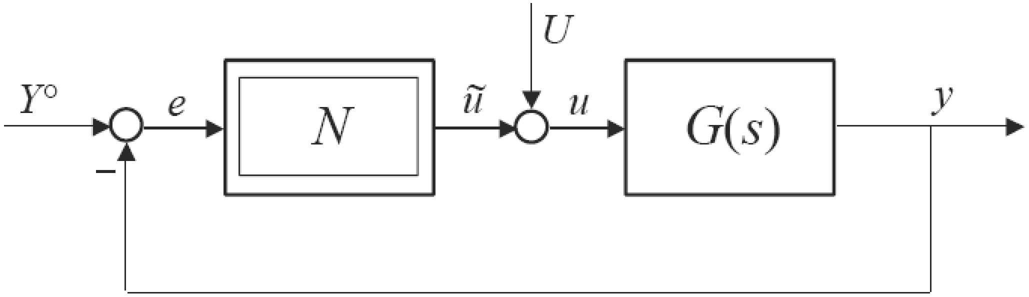
LIMIT CYCLES

Goals:

Provide conditions to assess

- existence
- amplitude
- stability

of periodic solutions in a time-invariant Lur’e system subject to constant inputs



STABILITY OF A PERIODIC SOLUTION

What is it meant by stable periodic solution?

A periodic solution is stable if, when the initial state is slightly perturbed, then the resulting evolution differs of a small amount from the periodic solution and tends to converge to it, with possibly a temporal mismatch (stability of the trajectory)

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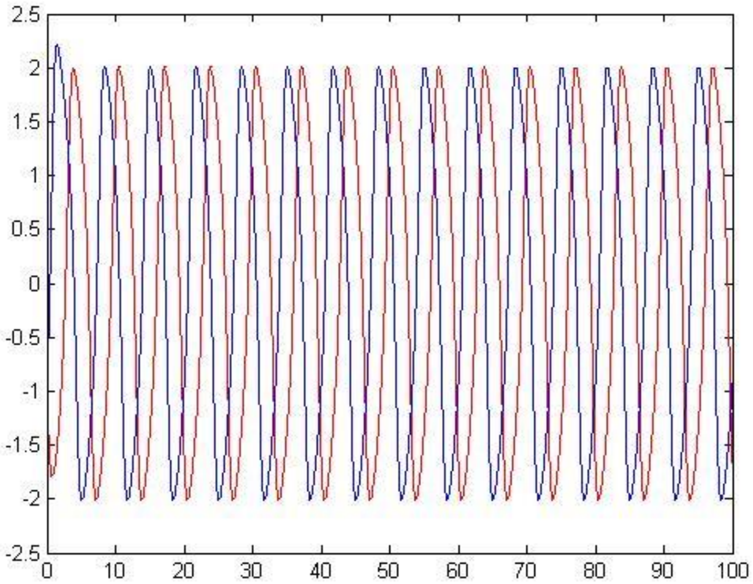
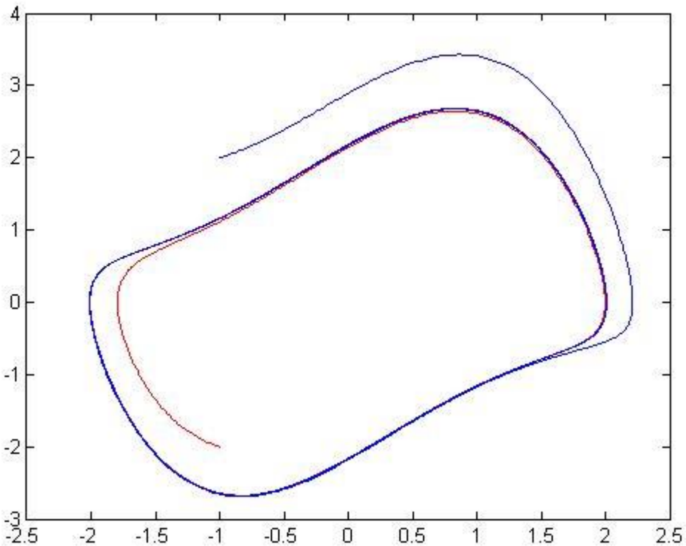




Remark:

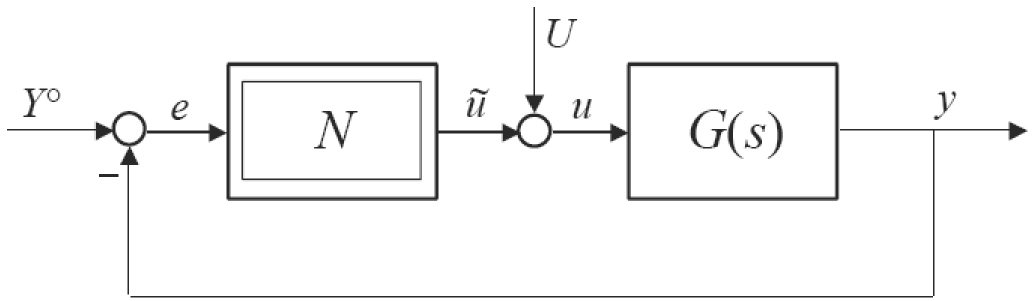
Depending on the fact that the limit cycle is

- desired (switching control)
- undesired (due to nonlinearity that were neglected at the control design stage)

we would like it to be stable or unstable

Assessing stability of a limit cycle is difficult and here we shall describe a heuristic method

STABILITY OF A PERIODIC SOLUTION



<u>Assumption:</u>

- *N* described by an input-output map
- $E_0 \cong 0 \quad \Rightarrow \quad e(t) \cong E\,cos(\Omega\,t)$

Let (E*,Ω*) be a solution to the harmonic balance equation

$$G(j\Omega) = \frac{-1}{D(E)} := H(E)$$

with associated the periodic solution: *e(t) = E*cos(Ω*t)*

CAHEN-LOEB CRITERION

Let (E^*,Ω^*) be a solution to the harmonic balance equation

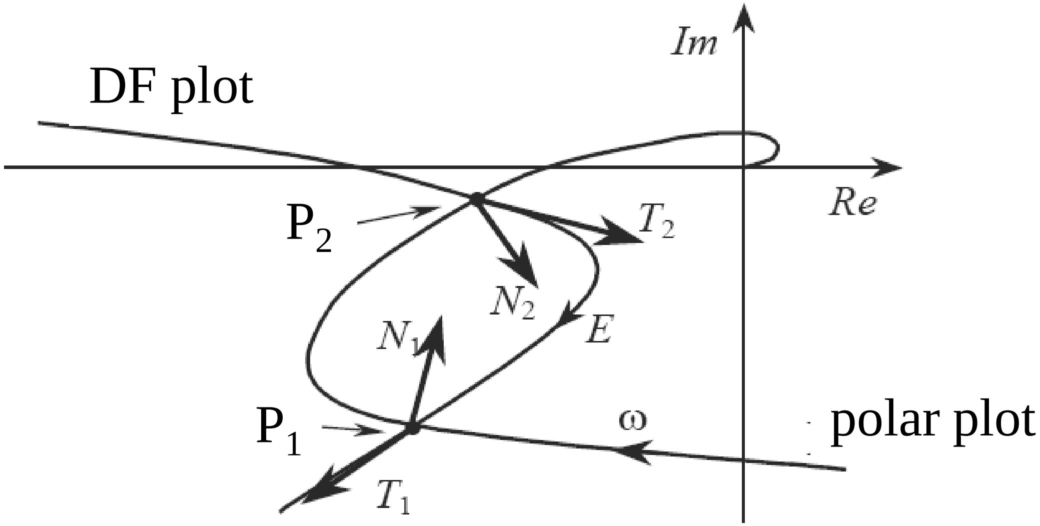
$$G(j\Omega) = \frac{-1}{D(E)} := H(E)$$

and P the corresponding intersection point in the graphical interpretation of the equation.

Let **T** be the vector tangent to the DF plot in P, pointing towards the direction where E increases.

Let **N** be a vector normal to the polar plot of G(s), pointing towards the right-hand-side when following the polar plot in the direction of the increasing angular frequency.

CAHEN-LOEB CRITERION



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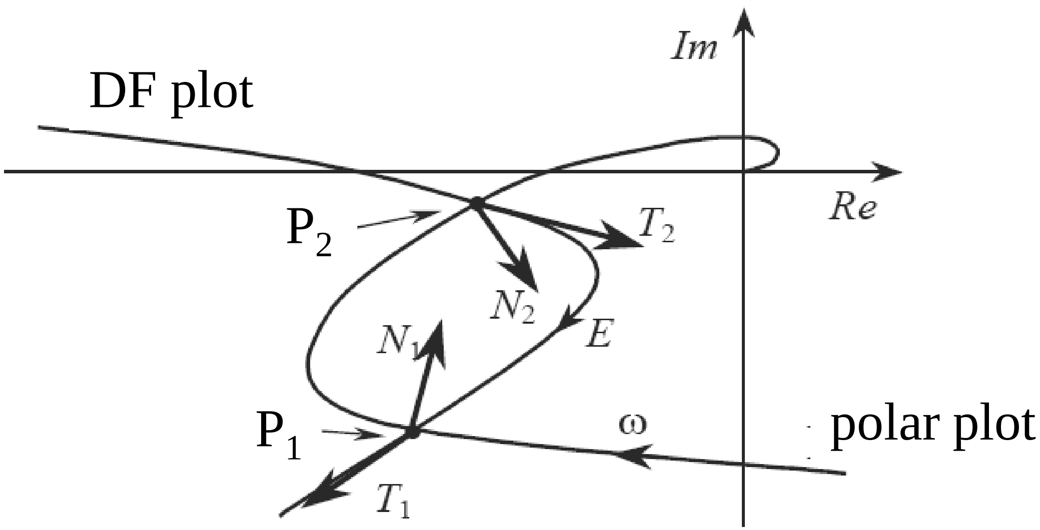

The periodic solution associated with P is stable if

$$T \times N < 0$$

unstable otherwise, where “x” denotes the scalar product.

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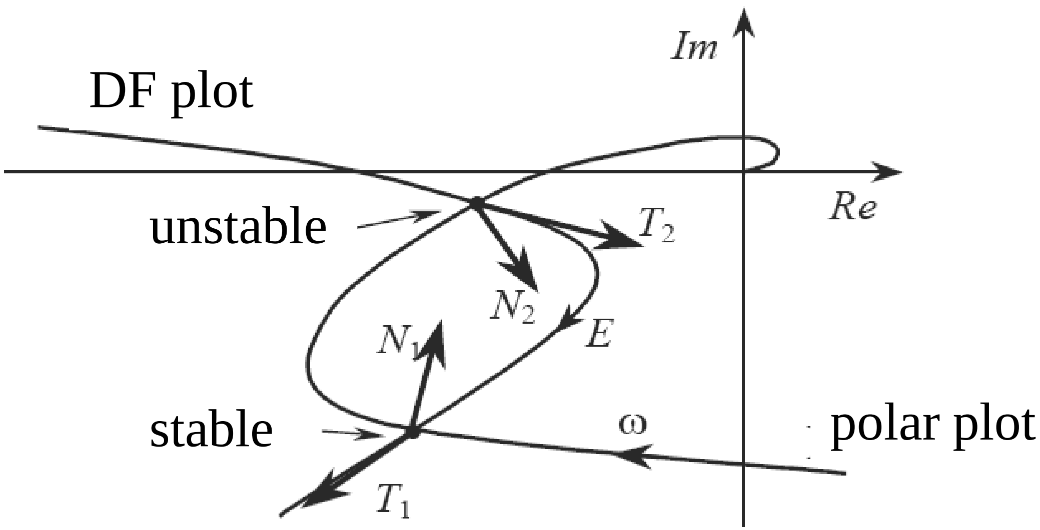

The periodic solution associated with P is stable if

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DESCRIBING FUNCTION METHOD

It is a heuristic method, since it is based on the filtering assumption

→ if the harmonic balance equation has a solution, then, a periodic solution with angular frequency and amplitude as given by the corresponding *e(t)* may be present

→ It might be the case that the predicted periodic solutions are not present, and also that there exist periodic solutions while the method does not predict any