INPUT-OUTPUT APPROACH

EXAMPLE: LINEAR ASYMPTOTICALLY STABLE DYNAMICAL SYSTEM

$$S: \begin{cases} \dot{x} = Ax + Bu, \quad x(0) = x_0 \in \Re^n \\ y = Cx \end{cases}$$

$$Re[\lambda_i(A)] < 0, \ i = 1, 2, \ldots, n$$

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$$S : \quad y(\cdot) = H(u(\cdot)) = G(u(\cdot)) + y_0(\cdot) \in \mathcal{L}_e, \ \forall u(\cdot) \in \mathcal{L}_e$$

$$y_0(t) = Ce^{At}x_0, \, t \in \Re^+$$

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The operator H is:

- causal
- biased $y_0(\cdot) \neq 0$
- G is the unbiased operator associated with H

$$\|G\| := \sup_{u \in \mathcal{L} \setminus \{0\}} \frac{\|G(u(\cdot))\|}{\|u(\cdot)\|} = \sup_{\|u(\cdot)\|=1} \frac{\|G(u(\cdot))\|}{\|u(\cdot)\|}$$

induced norm of G

G is a linear operator

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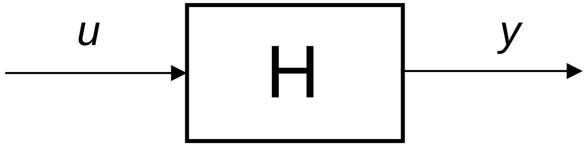
Let $\mathcal{L} = L_\infty$

Is H bounded?

WEAKLY BOUNDED/BOUNDED OPERATOR



$H : \mathcal{L}_e \to \mathcal{L}_e$ causal

Definition (weakly bounded operator):

A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is weakly bounded (or with finite gain) if

$$\exists\ \hat{\gamma}, \hat{\beta} \in \Re^+ :\ \|H(u(\cdot))\| \leq \hat{\gamma}\|u(\cdot)\| + \hat{\beta},\ \forall u(\cdot) \in \mathcal{L}$$

Definition (bounded operator):

A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is bounded if

- $\|y_0(\cdot)\| = \|H(0)\| := \beta < \infty\ (\Leftrightarrow y_0(\cdot) \in \mathcal{L})$
- $G(\cdot) = H(\cdot) - H(0)$ is bounded

Remark: H bounded → H weakly bounded

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Let $\mathcal{L} = L_\infty$

Is H bounded?

- $y_0(t) = Ce^{At}x_0, \, t \in \Re^+$ free evolution of S

$$\sup_{t\in\Re^+} |y_0(t)| < \infty \rightarrow y_0(\cdot) \in L_\infty \rightarrow \beta := \|y_0(\cdot)\|_\infty$$

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G bounded → H bounded

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Is G bounded?

$$\begin{aligned}\|G\|_\infty &= \sup_{\|u(\cdot)\|_\infty = 1} \|G(u(\cdot))\|_\infty = \sup_{\|u(\cdot)\|_\infty = 1} \sup_{t \geq 0} \left| \int_0^t g(t-\tau)u(\tau)d\tau \right| \\ &\leq \sup_{\|u(\cdot)\|_\infty = 1} \sup_{t \geq 0} \int_0^t |g(t-\tau)||u(\tau)|d\tau \leq \sup_{t \geq 0} \int_0^t |g(t-\tau)|d\tau\end{aligned}$$

$$\|u(\cdot)\|_\infty = 1 \rightarrow |u(t)| \leq 1, \forall t \geq 0$$

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$$\int_0^\infty |g(t)|dt < \infty \rightarrow g(\cdot) \in L_1 \rightarrow k_1 := \|g(\cdot)\|_1$$

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H è limitato?

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→ Let us compute the zero bias gain of G $\gamma_\infty^\circ(G) = \|G\|_\infty$

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- $\gamma_\infty^\circ(G) = \|G\|_\infty = \sup_{\|u(\cdot)\|_\infty = 1} \|G(u(\cdot))\| \leq k_1$

- If we find a sequence of inputs u_m, with $\|u_m(\cdot)\|_\infty = 1$ such that

$$\lim_{m \to +\infty} \|G(u_m(\cdot))\|_\infty = \int_0^\infty |g(t-\tau)|d\tau = k_1$$

→ G bounded operator in $L_{\infty e}$ with zero bias gain

$$\gamma_\infty^\circ(G) = k_1$$

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Let $\mathcal{L} = L_\infty$

Let $m \in \mathbb{N}$ and define

$$u_m(t) := \begin{cases} sgn(g(m-t)), & 0 \leq t \leq m \\ 0, & t > m \end{cases}$$

$$\|u_m(\cdot)\|_\infty = 1$$

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H (bounded) is weakly bounded because

$$\begin{aligned} \|y(\cdot)\|_\infty &= \|G(u(\cdot)) + y_0(\cdot)\|_\infty \leq \|G(u(\cdot))\|_\infty + \|y_0(\cdot)\|_\infty \\ &\leq k_1 \|u(\cdot)\|_\infty + \beta, \ \forall u(\cdot) \in L_\infty \end{aligned}$$

$$\gamma_\infty(H) \leq \gamma_\infty^\circ(G) = k_1$$

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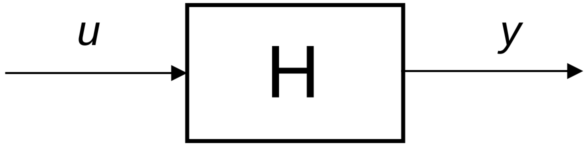
$$\gamma_\infty(H) \leq \gamma_\infty^\circ(G) = k_1$$

We shall show that $\gamma_\infty(H) = \gamma_\infty^\circ(G) = k_1 = \|g(\cdot)\|_1$

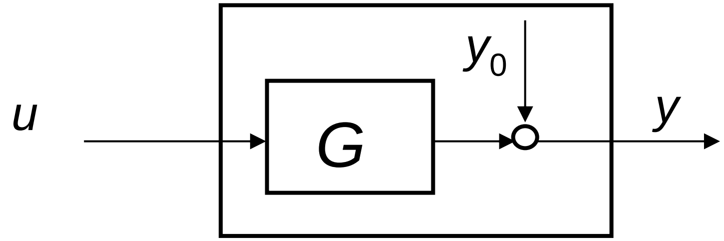
AFFINE CAUSAL OPERATOR



$H : \mathcal{L}_e \to \mathcal{L}_e$ causal operator

Definition (affine operator):

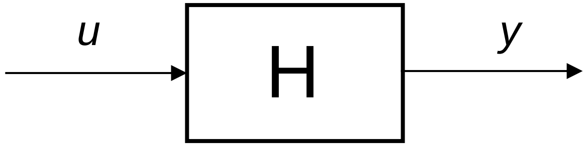
The causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is affine if the associated unbiased operator *G*



$$G : \mathcal{L}_e \to \mathcal{L}_e \qquad u(\cdot) \in \mathcal{L}_e \to G(u(\cdot)) = H(u(\cdot)) - y_0(\cdot) \in \mathcal{L}_e$$

Is linear.

AFFINE CAUSAL OPERATOR



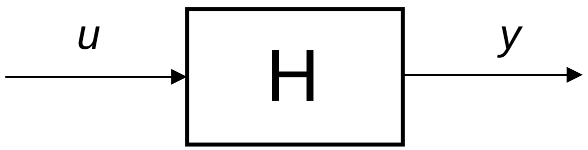
$H : \mathcal{L}_e \to \mathcal{L}_e$ affine causal operator

Theorem

A weakly bounded affine causal operator

- is bounded
- its gain is equal to its zero bias gain

AFFINE CAUSAL OPERATOR



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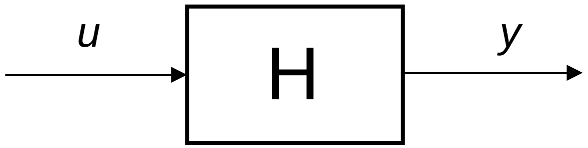
Proof:

H weakly bounded causal → $\exists\ \gamma(H), \beta \in \Re^+$ such that

$$\|H(u(\cdot))\| \leq \gamma(H)\|u(\cdot)\| + \beta, \quad \forall u(\cdot) \in \mathcal{L}$$

→ $\|y_0(\cdot)\| = \|H(0)\| \leq \beta < \infty$ (first condition for H to be bounded)

AFFINE CAUSAL OPERATOR



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<u>Proof:</u>

$$\|H(u(\cdot))\| \leq \gamma(H)\|u(\cdot)\| + \beta, \quad \forall u(\cdot) \in \mathcal{L}$$

For any $u(\cdot) \in \mathcal{L} \setminus \{0\}$ & $\alpha > 0$

$$\lim_{\alpha\rightarrow\infty} \frac{\|H(\alpha u(\cdot))\|}{\|\alpha u(\cdot)\|} \leq \lim_{\alpha\rightarrow\infty} \frac{\gamma(H)\|\alpha u(\cdot)\| + \beta}{\|\alpha u(\cdot)\|} = \gamma(H) + \lim_{\alpha\rightarrow\infty} \frac{\beta}{\alpha\|u(\cdot)\|} = \gamma(H)$$

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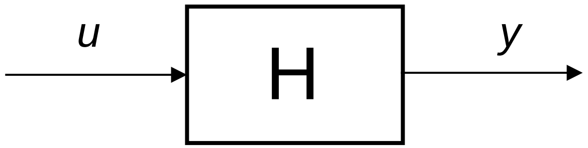

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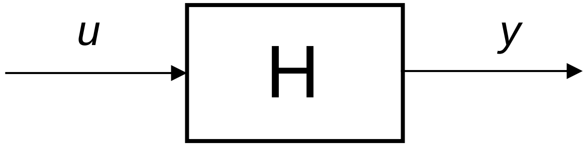
Proof:

$$a, b \in \mathcal{L}, c = a + b \Rightarrow \|a\| - \|b\| \leq \|c\| \leq \|a\| + \|b\|$$

$$a = c - b \Rightarrow \|a\| \leq \|b\| + \|c\|$$

AFFINE CAUSAL OPERATOR



$H : \mathcal{L}_e \to \mathcal{L}_e$ affine causal operator

Theorem

A weakly bounded affine causal operator

- is bounded
- its gain is equal to its zero bias gain

Proof:

$$\frac{\|G(\alpha u(\cdot))\| - \|y_0(\cdot)\|}{\|\alpha u(\cdot)\|} \leq \frac{\|H(\alpha u(\cdot))\|}{\|\alpha u(\cdot)\|} \leq \frac{\|G(\alpha u(\cdot))\| + \|y_0(\cdot)\|}{\|\alpha u(\cdot)\|}$$

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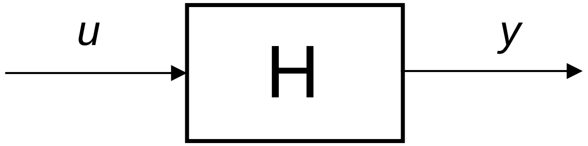

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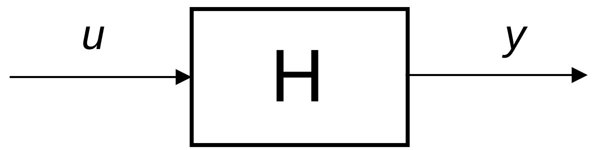

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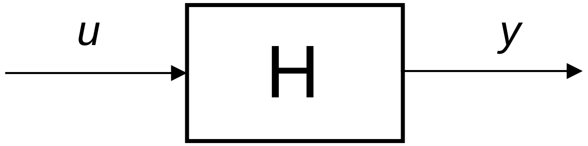

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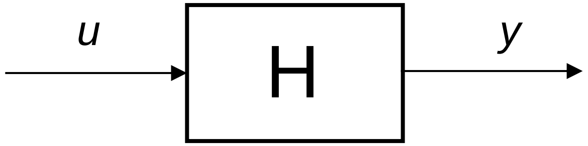
G bounded (second condition for H to be bounded)

AFFINE CAUSAL OPERATOR



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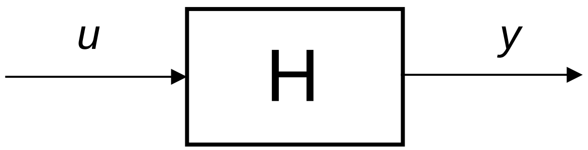
G bounded (second condition for H to be bounded) → H bounded

AFFINE CAUSAL OPERATOR



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G bounded (→ H bounded) with

$$\gamma^\circ(G) = \sup_{u(\cdot)\in\mathcal{L}\setminus\{0\}} \frac{\|G(u(\cdot))\|}{\|u(\cdot)\|} \leq \gamma(H)$$

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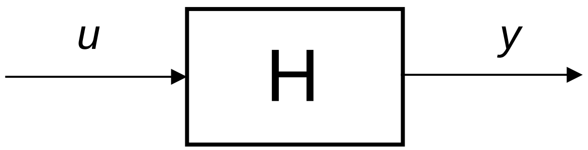

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$$\gamma(H) \leq \gamma^\circ(G)$$

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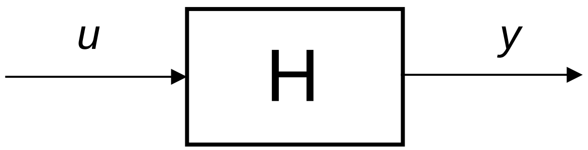

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$$S: \quad y(\cdot) = H(u(\cdot)) = G(u(\cdot)) + y_0(\cdot) \in \mathcal{L}_e, \;\; \forall u(\cdot) \in \mathcal{L}_e$$

$$y_0(t) = Ce^{At}x_0, \; t \in \Re^+$$

$$G(u(\cdot)) := \int_0^t g(t-\tau)u(\tau)d\tau = g(\cdot) * u(\cdot)$$

Let $\mathcal{L} = L_\infty$

H (bounded) is weakly bounded because

$$\begin{aligned} \|y(\cdot)\|_\infty \quad &= \|G(u(\cdot)) + y_0(\cdot)\|_\infty \leq \|G(u(\cdot))\|_\infty + \|y_0(\cdot)\|_\infty \\ &\leq k_1\|u(\cdot)\|_\infty + \beta, \;\; \forall u(\cdot) \in L_\infty \end{aligned}$$

$$\gamma_\infty(H) \leq \gamma_\infty^\circ(G) = k_1$$

$\gamma_\infty(H) = \gamma_\infty^\circ(G) = k_1 = \|g(\cdot)\|_1$ since H is affine

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Is H bounded?

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- $y_0(t) = Ce^{At}x_0, \ t \in \Re^+$ free evolution of S

$$\left(\int_0^\infty |Ce^{At}x_0|^2 dt \right)^{\frac{1}{2}} < \infty \rightarrow y_0(\cdot) \in L_2 \rightarrow \beta := \|y_0(\cdot)\|_2$$

If G bounded
→ H bounded (and, hence, weakly bounded) and, since H is affine, its gain is equal to the zero bias gain, i.e., $\gamma_2(H) = \gamma_2^\circ(G)$

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$$z(\cdot) = G(u(\cdot)) \ \leftrightarrow \ Z(j\omega) = F(j\omega)U(j\omega), \quad \forall u \in L_2$$

Fourier transform of the impulse response:

$$F(j\omega) = C(j\omega)(j\omega I - A)^{-1}B$$

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By Parseval theorem:

$$\|z(\cdot)\|_2^2 = \int_0^\infty z^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |Z(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(j\omega)|^2 |U(j\omega)|^2 d\omega$$

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$$\leq F_{\max}^2 \frac{1}{2\pi}\int_{-\infty}^{+\infty} |U(j\omega)|^2 d\omega = F_{\max}^2 \int_0^\infty u^2(t)dt = F_{\max}^2 \|u(\cdot)\|_2^2$$

$$F_{\max} := \max_{\omega \in \Re^+} |F(j\omega)| < \infty$$

since it is a continuous function that tends to zero as $\omega \to \infty$

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Let $\mathcal{L} = L_2$

Is G bounded?

$$\|G(u(\cdot))\|_2 = \|z(\cdot)\|_2 \leq F_{\max}\|u(\cdot)\|_2, \ \forall u(\cdot) \in L_2$$

from which we get

$$\|G\|_2 := \sup_{\|u(\cdot)\|_2=1} \|G(u(\cdot))\|_2 \leq F_{\max}$$

→ G is bounded

One can show that

$$\|G\|_2 = F_{\max} = \max_{\omega \in \Re^+} |F(j\omega)|$$

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Let $\mathcal{L} = L_2$

In conclusion:

H is bounded and weakly bounded with gain

$$\gamma_2(H) = \gamma_2^\circ(G) = \max_{\omega \in \Re^+} |F(j\omega)|$$

H_∞ norm of F(s)

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One can show that the *H* operator is bounded (and, hence, weakly bounded, with gain equal to the zero bias gain) in L_{pe}, for any p