INPUT-OUTPUT APPROACH: STABILITY

INPUT-OUTPUT STABILITY

Definition ($\mathcal{L}$ stability):

A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$ - stable if $H(\mathcal{L}) \subseteq \mathcal{L}$, that is

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- If $\mathcal{L} = L_\infty \to$ BIBO (bounded input bounded output) stability

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Remarks

- It is a property of the system
- It applies to both static and dynamic systems
- It depends on $\mathcal{L}$

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A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$ - stable if and only if there exist

- a continuous increasing function $\sigma(\cdot) : \Re^+ \to \Re^+$ with $\sigma(0) = 0$
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<u>Proof.</u>

← straightforward

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<u>Proof.</u> (→)

If H is $\mathcal{L}$ - stable, then for any $v \in \Re^+$ $\zeta(v) := \sup_{\|u(\cdot)\| \leq v, u(\cdot) \in \mathcal{L}} \|H(u(\cdot))\|$
is well-defined and finite, from which we get

$$\|H(u(\cdot))\| \leq \zeta(\|u(\cdot)\|), \ \forall u(\cdot) \in \mathcal{L}$$

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Proof. (→)

Since $\zeta(\cdot) : \Re^+ \to \Re^+$ is a non negative function that is non decreasing, then, there exists a function $\sigma(\cdot) : \Re^+ \to \Re^+$ continuous and increasing with $\sigma(0) = 0$ and $\beta \in \Re^+$ such that

$$\zeta(v) \leq \sigma(v) + \beta, \ \forall v \in \Re^+$$

and, hence,

$$\|H(u(\cdot))\| \leq \zeta(\|u(\cdot)\|) \leq \sigma(\|u(\cdot)\|) + \beta, \ \forall u(\cdot) \in \mathcal{L}$$

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'finite gain $\mathcal{L}$ - stability'

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<u>Corollary</u>

A causal weakly bounded operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is $\mathcal{L}$ - stable.

Remark:
the opposite is not true, in general (example: $\mathcal{L} = L_\infty$ and static system described by a continuous function that grows more than linearly)

INPUT-OUTPUT AND LYAPUNOV STABILITY

Problem:

Identify connections between various kinds of I/O stability and of Lyapunov stability.

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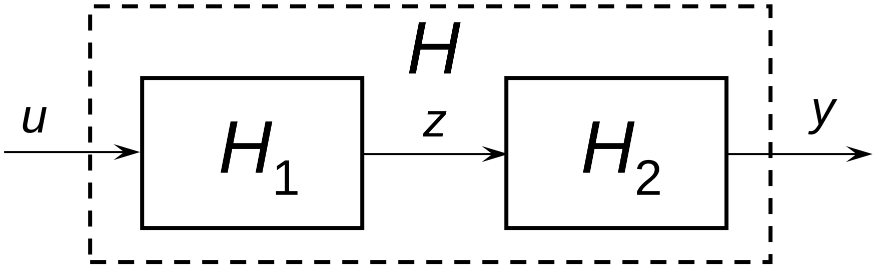
S asymptotically stable → the operator H associated with S is L_p-stable for any $p \in (0,\infty]$

H is L_p-stable, $p \in (0,\infty]$ → S is asymptotically stable if and only if ts non-observable and non-reachable parts are asymptotically stable

STABILITY OF INTECONNECTED SYSTEMS: CASCADE



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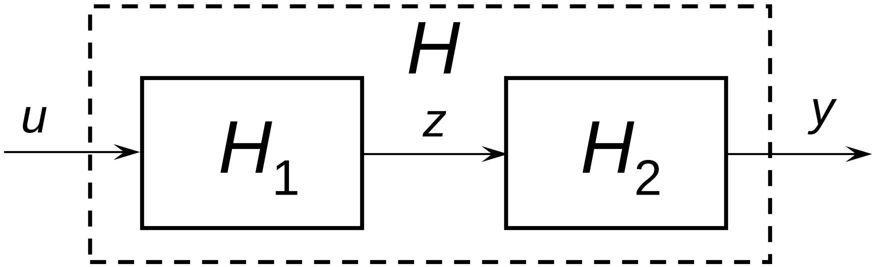

$$H_i : \mathcal{L}_e \to \mathcal{L}_e, \; i = 1, 2$$

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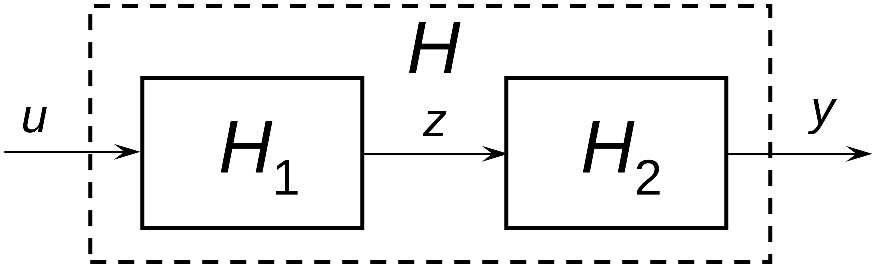

$$H_i : \mathcal{L}_e \to \mathcal{L}_e, \;\; i = 1, 2$$

Theorem

Two causal and weakly bounded operators H_1 e H_2, interconnected in cascade, originates an operator H

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causal and weakly bounded with gain $\gamma(H) \leq \gamma(H_1)\gamma(H_2)$

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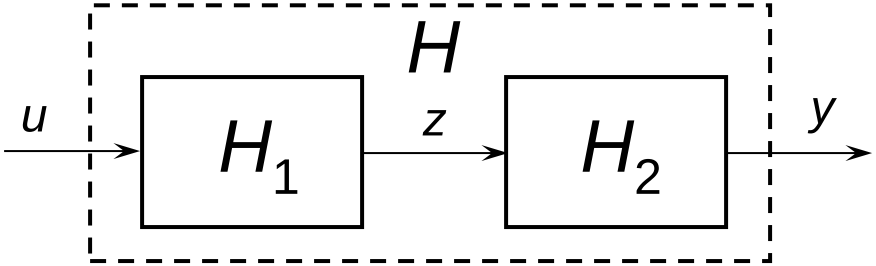

$$H_i : \mathcal{L}_e \to \mathcal{L}_e,\ \ i = 1, 2$$

<u>Proof:</u>

H_1 weakly bounded implies that

$$\exists\, \gamma_1, \beta_1 \in \Re^+ :\ \ \|H_1(u(\cdot))\| \le \gamma_1 \|u(\cdot)\| + \beta_1, \forall u(\cdot) \in \mathcal{L} \to z(\cdot) = H_1(u(\cdot)) \in \mathcal{L}$$

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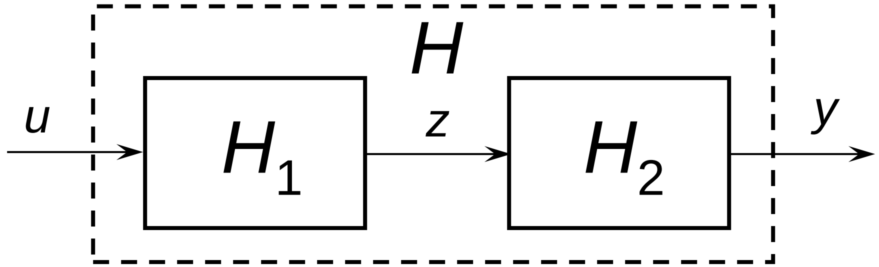

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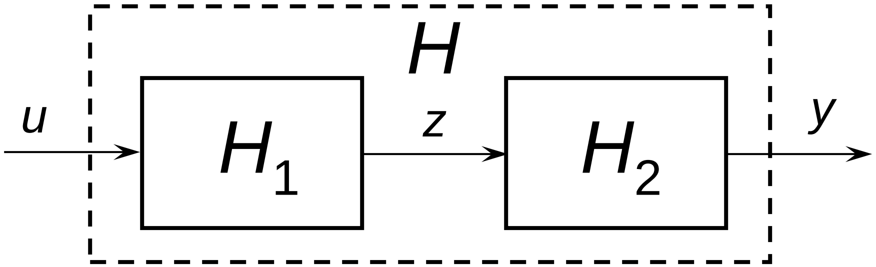

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Then,

$$\begin{aligned} \|H(u(\cdot))\| &= \|H_2(H_1(u(\cdot)))\| \le \gamma_2(\gamma_1 \|u(\cdot)\| + \beta_1) + \beta_2 \\ &= \gamma_2\gamma_1 \|u(\cdot)\| + \gamma_2\beta_1 + \beta_2, \forall u(\cdot) \in \mathcal{L} \end{aligned}$$

that is H is weakly bounded and $\gamma(H) \le \gamma(H_1)\gamma(H_2)$

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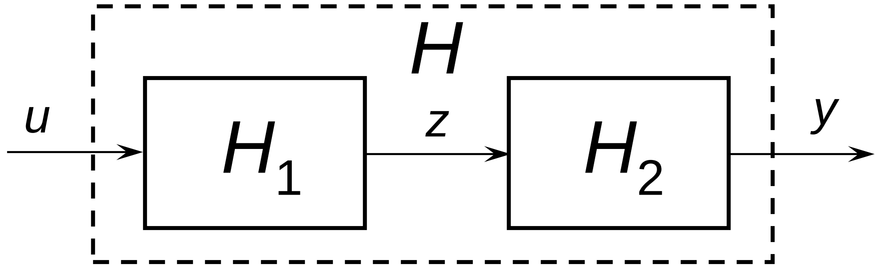


$$H_i : \mathcal{L}_e \to \mathcal{L}_e,\ \ i = 1, 2$$

Example:

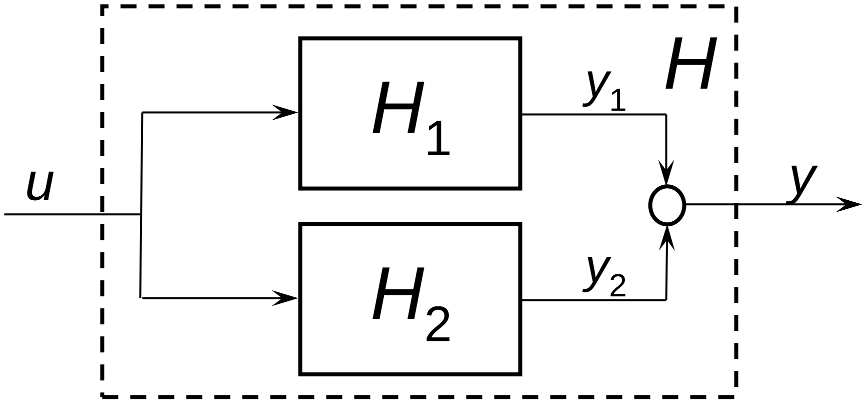
Linear asymptotically stable time invariant dynamical systems with transfer functions $F_1(s)$ and $F_2(s)$

→ The cascade system has transfer function $F(s) = F_1(s)F_2(s)$

Let $\mathcal{L} = L_2$. Then,

$$\gamma_2(H) = F_{\max} = \max_{\omega \in \Re^+} |F(j\omega)| \leq F_{1,\max} F_{2,\max} = \gamma_2(H_1)\gamma_2(H_2)$$

STABILITY OF INTECONNECTED SYSTEMS: PARALLEL



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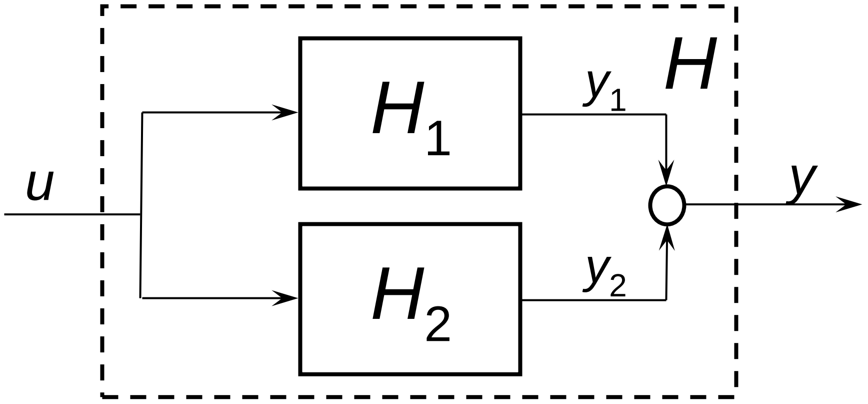

$$H_i : \mathcal{L}_e \rightarrow \mathcal{L}_e,\ \ i = 1, 2$$

$$u(\cdot) \in \mathcal{L}_e \rightarrow y(\cdot) = H_1(u(\cdot)) + H_2(u(\cdot)) \in \mathcal{L}_e$$

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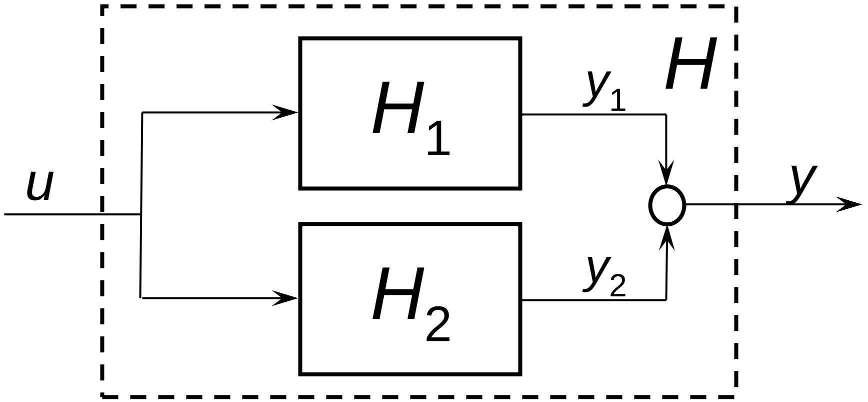

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Two causal and weakly bounded operators H_1 e H_2, interconnected in parallel, originates an operator H

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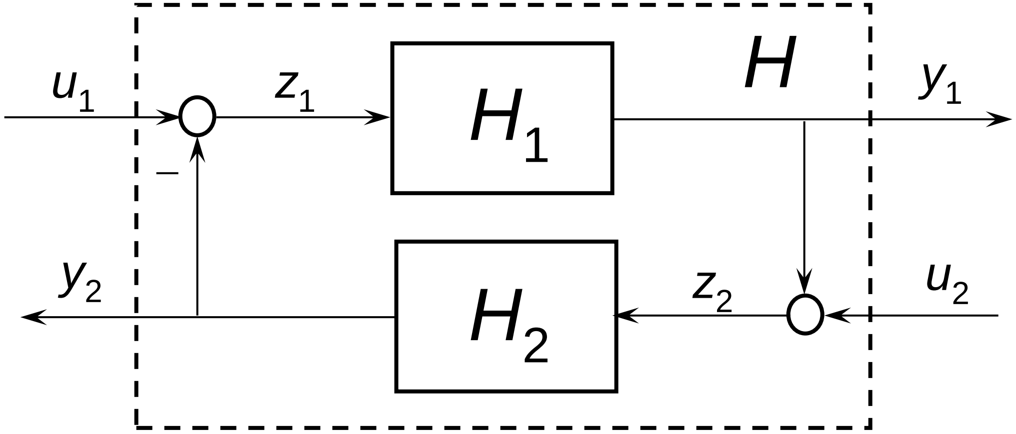
causal and weakly bounded with gain $\gamma(H) \leq \gamma(H_1) + \gamma(H_2)$

<u>Proof: [to do as exercise]</u>

STABILITY OF INTERCONNECTED SYSTEMS: FEEDBACK



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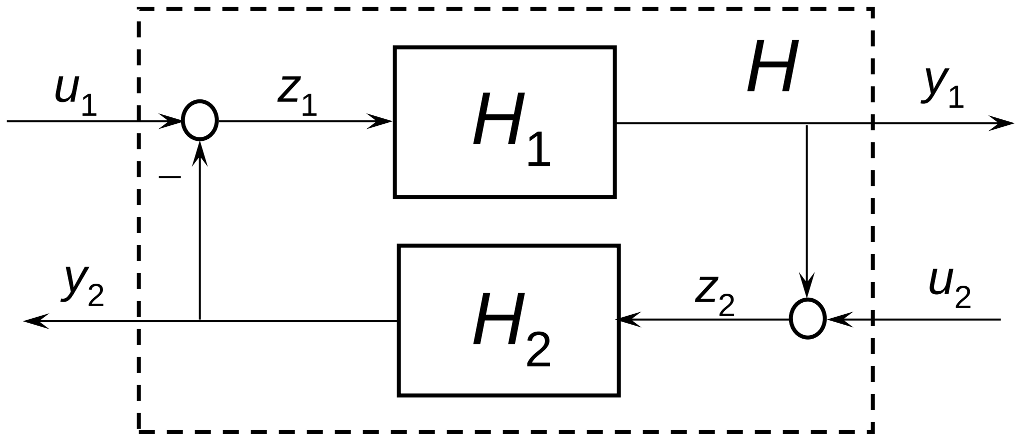

$$H_i : \mathcal{L}_e \to \mathcal{L}_e,\ \ i = 1, 2$$

- Is the operator *H* obtained by interconnecting in feedback the causal operators H_1 and H_2 is well-posed, i.e., the pair (y_1, y_2) exists and is unique for any $(u_1, u_2) \in \mathcal{L}_e \times \mathcal{L}_e$?

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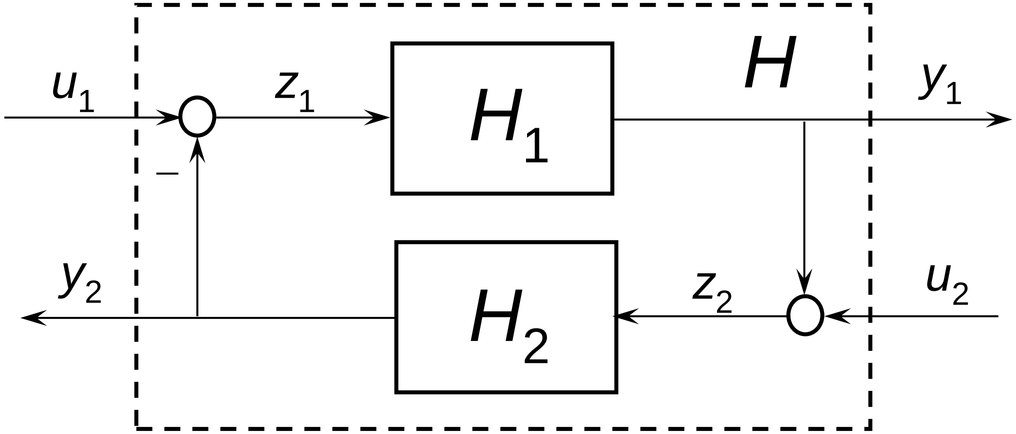

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  No, in general… It is well-posed if one of the two causal operators is strictly proper.

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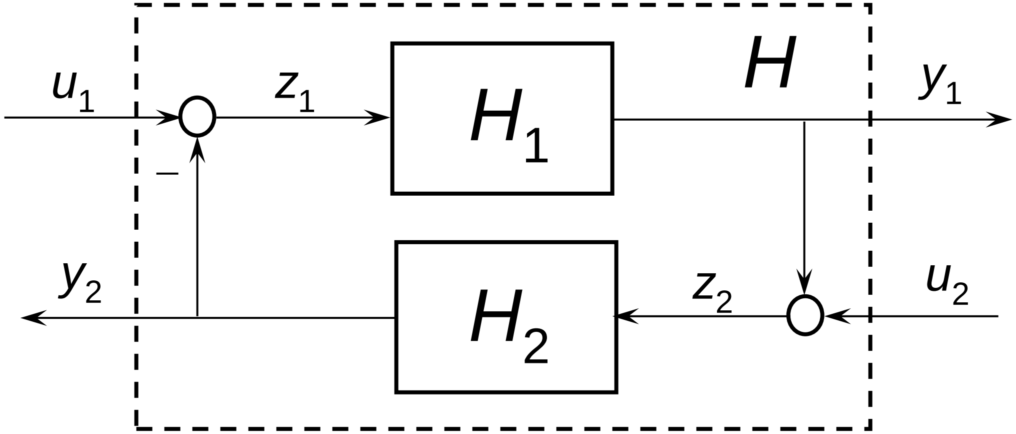

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- The operator *H* has two inputs and two outputs. Let us define the operators with one input and one output:

$$H_{ij} : \mathcal{L}_e \to \mathcal{L}_e \quad y_i(\cdot) = H_{ij}(u_j(\cdot)), \ i, j = 1, 2$$

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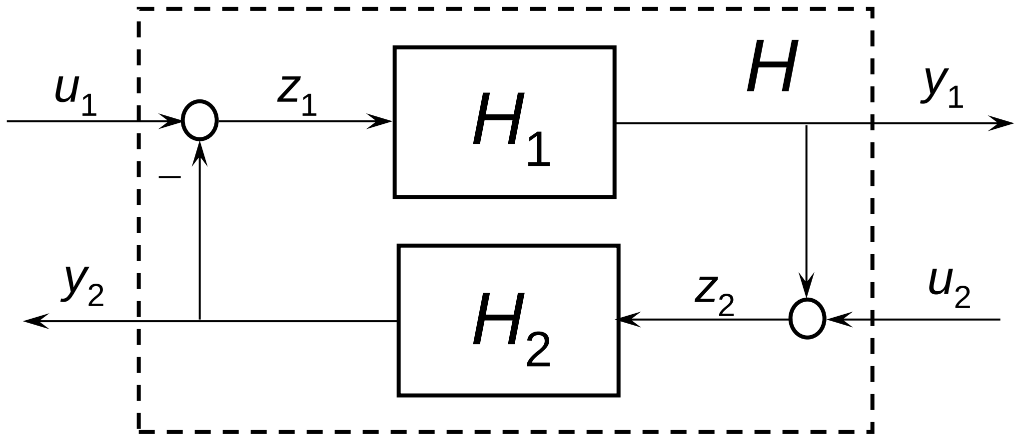

Small gain theorem

Let *H* be a well-posed causal operator obtained by connecting in feedback two causal and weakly bounded operators H_1 and H_2. If

$$\lambda := \gamma(H_1)\gamma(H_2) < 1$$

then, *H* is weakly bounded, that is:

$$\exists\ \hat{\gamma}_{i1}, \hat{\gamma}_{i2}, \hat{\beta}_i \in\ Re^+ :\ \|y_i(\cdot)\| \leq \hat{\gamma}_{i1}\|u_1(\cdot)\| + \hat{\gamma}_{i2}\|u_2(\cdot)\| + \hat{\beta}_i$$

$$\forall u_1(\cdot), u_2(\cdot) \in \mathcal{L}, i = 1, 2$$

Furthermore, $\gamma(H_{11}) \leq \dfrac{\gamma(H_1)}{1-\lambda},\ \gamma(H_{22}) \leq \dfrac{\gamma(H_2)}{1-\lambda},\ \gamma(H_{12}), \gamma(H_{21}) \leq \dfrac{\lambda}{1-\lambda}$

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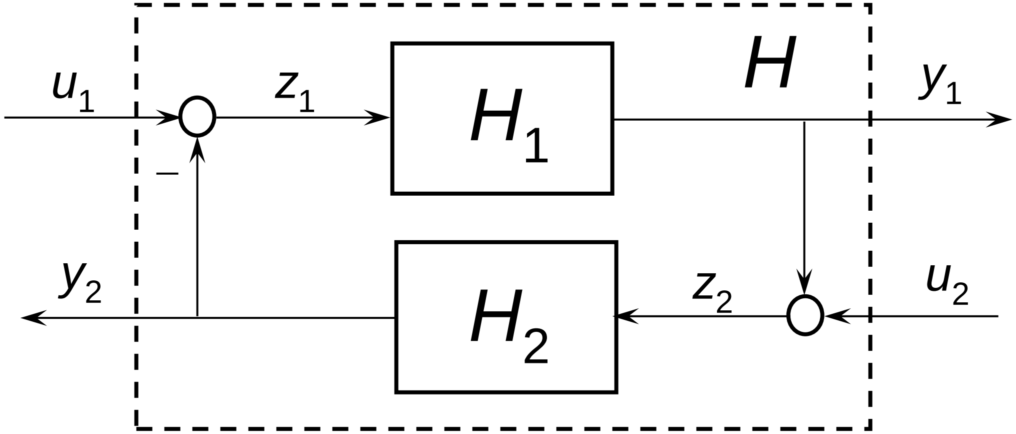

Proof (small gain theorem)

If H_1 e H_2 are causal weakly bounded and H is well-posed, then

$$\forall u_1(\cdot), u_2(\cdot), z_1(\cdot) = u_1(\cdot) - y_2(\cdot), z_2(\cdot) = u_2(\cdot) + y_1(\cdot) \in \mathcal{L}_e, \ \forall \tau \in \Re^+$$

we have that

$$\|y_{1\tau}(\cdot)\| \quad \leq \gamma_1 \|z_{1\tau}(\cdot)\| + \beta_1 \leq \gamma_1 (\|u_{1\tau}(\cdot)\| + \|y_{2\tau}(\cdot)\|) + \beta_1$$

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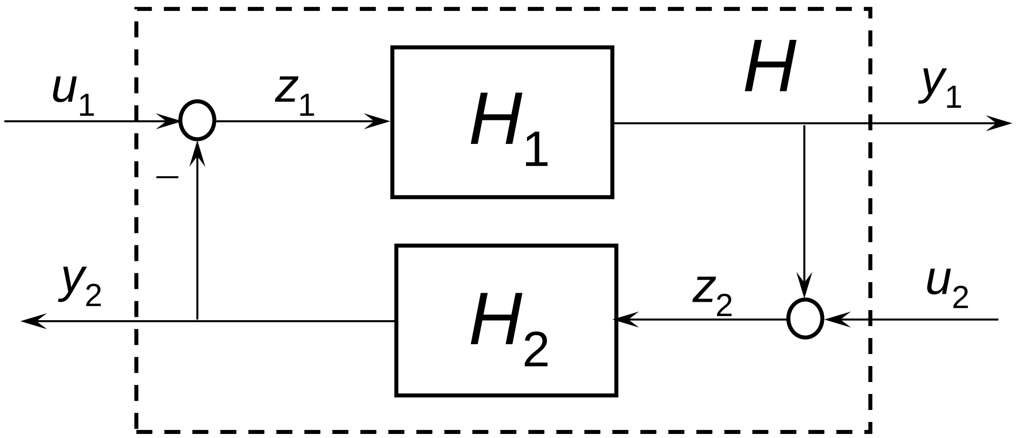

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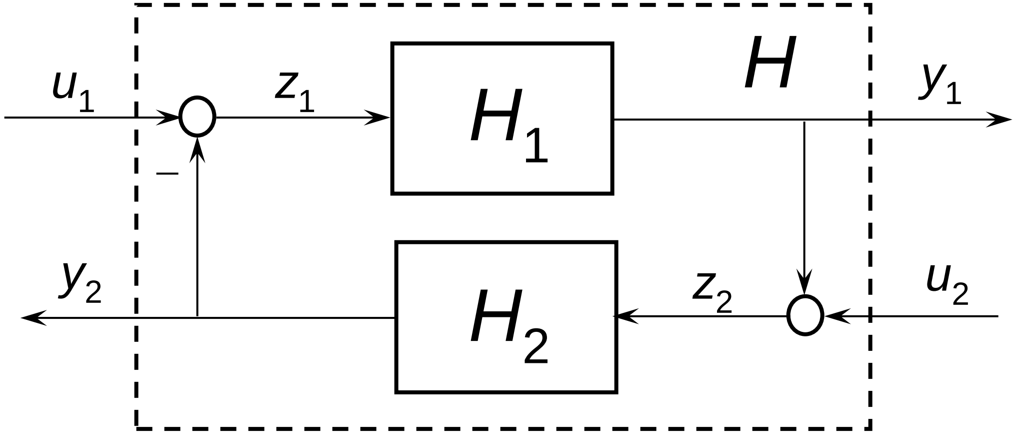

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$$\begin{aligned}
\|y_{1\tau}(\cdot)\| &\leq \gamma_1 \|z_{1\tau}(\cdot)\| + \beta_1 \leq \gamma_1 (\|u_{1\tau}(\cdot)\| + \|y_{2\tau}(\cdot)\|) + \beta_1 \\
&\leq \gamma_1 \|u_{1\tau}(\cdot)\| + \gamma_1 (\gamma_2 \|z_{2\tau}(\cdot)\| + \beta_2) + \beta_1 \\
&\leq \gamma_1 \|u_{1\tau}(\cdot)\| + \gamma_1 \gamma_2 (\|u_{2\tau}(\cdot)\| + \|y_{1\tau}(\cdot)\|) + \gamma_1 \beta_2 + \beta_1
\end{aligned}$$

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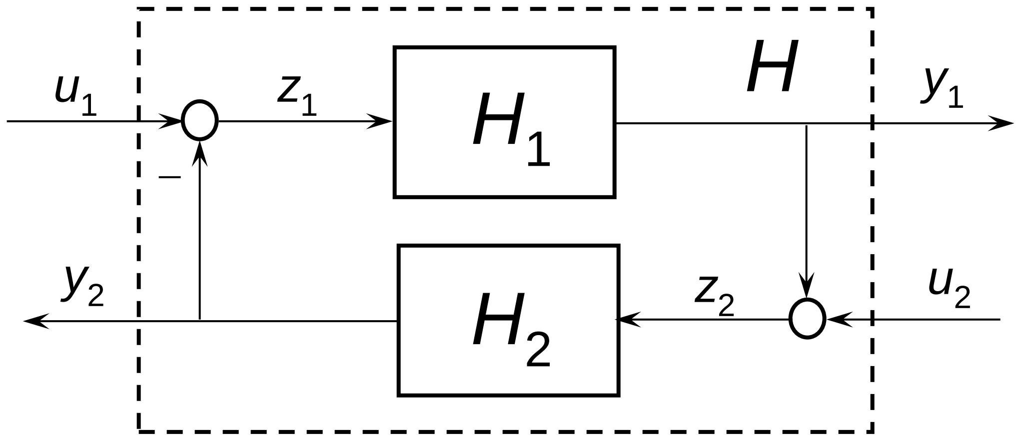

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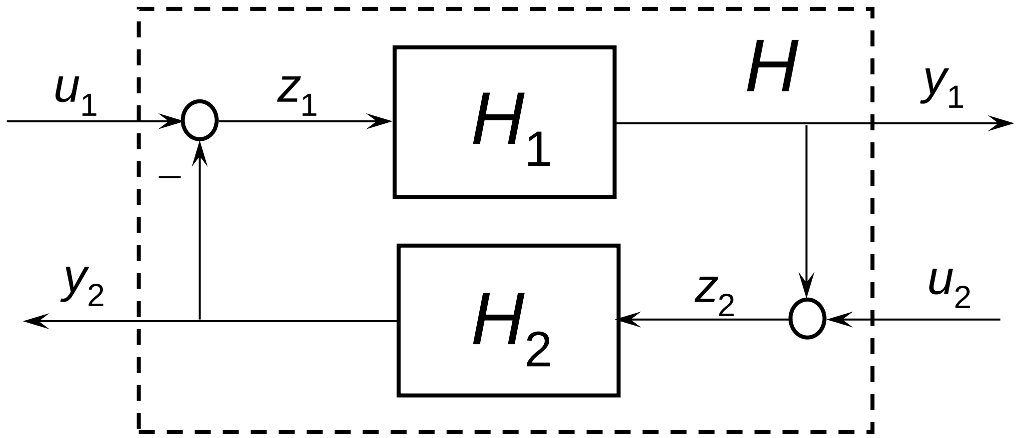
STABILITY OF INTERCONNECTED SYSTEMS: FEEDBACK



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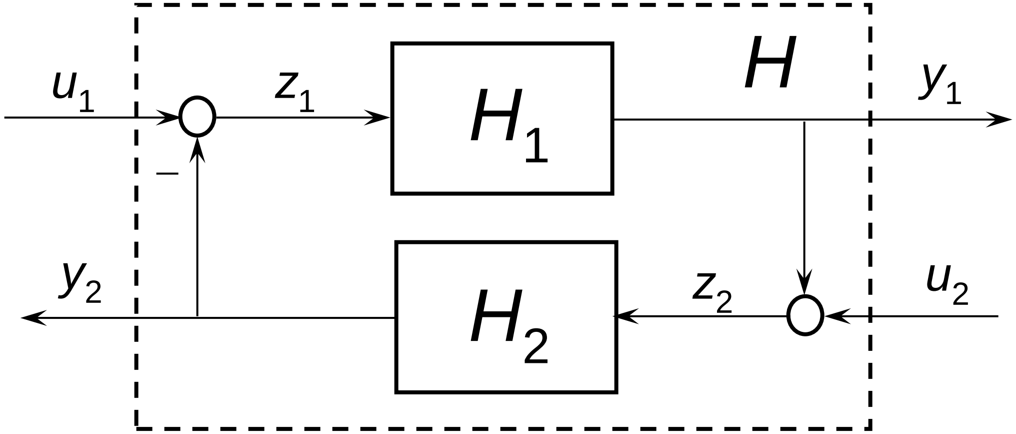

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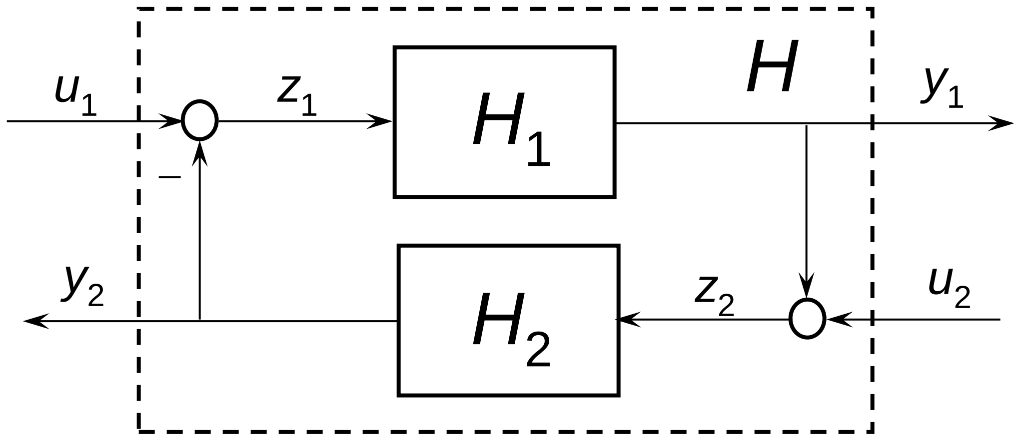

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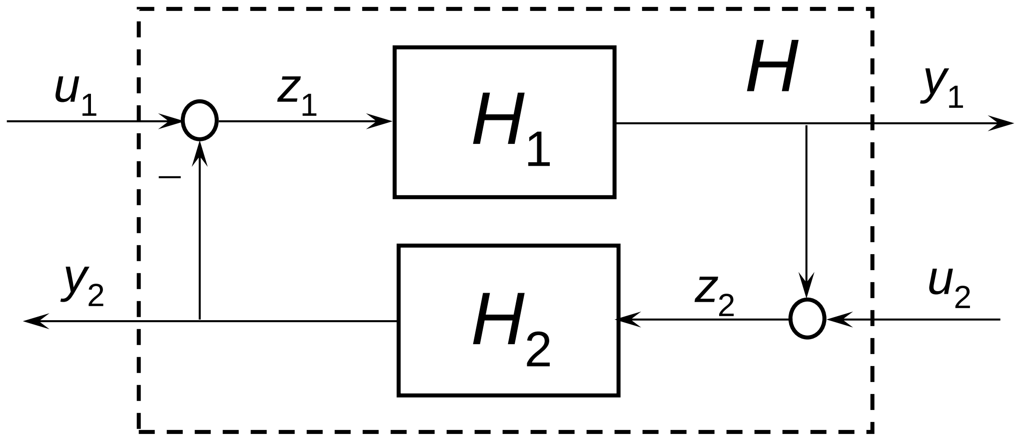

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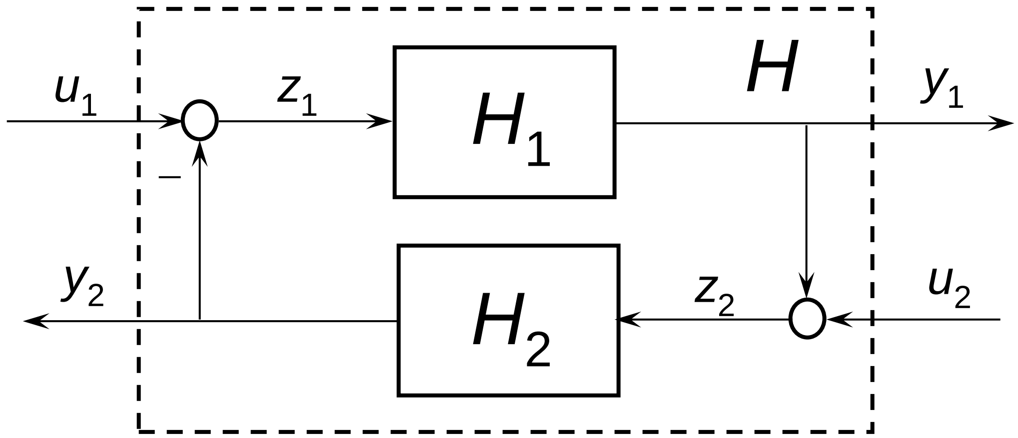
Let $f_1(\gamma_1, \gamma_2) := \frac{\gamma_1}{1-\gamma_1\gamma_2}, \ f_2(\gamma_1, \gamma_2) := \frac{\gamma_2}{1-\gamma_1\gamma_2}, f_{12}(\gamma_1, \gamma_2) := \frac{\gamma_1\gamma_2}{1-\gamma_1\gamma_2}$

STABILITY OF INTERCONNECTED SYSTEMS: FEEDBACK



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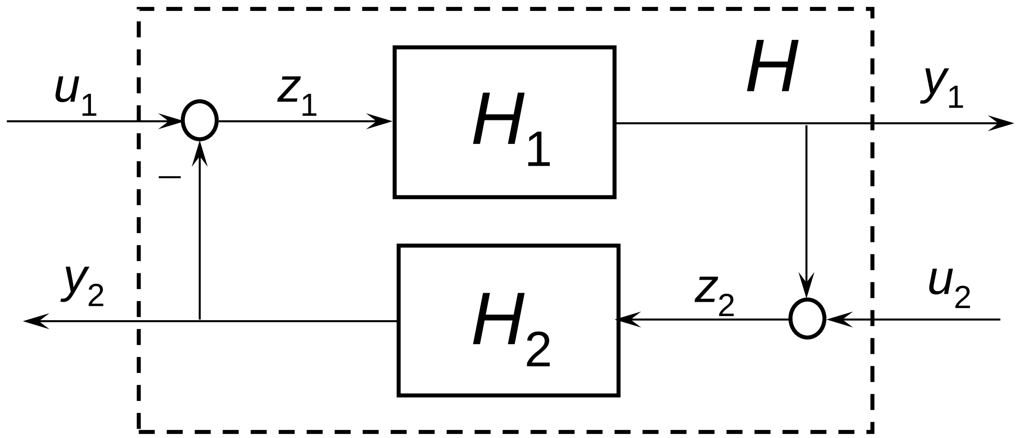
Are increasing function of γ_1 and γ_2 in the region where $\gamma_1\gamma_2 < 1$

→ $$\gamma(H_{11}) \leq \frac{\gamma(H_1)}{1-\lambda}, \; \gamma(H_{22}) \leq \frac{\gamma(H_2)}{1-\lambda}, \; \gamma(H_{12}), \gamma(H_{21}) \leq \frac{\lambda}{1-\lambda}$$

STABILITY OF INTERCONNECTED SYSTEMS: FEEDBACK



Small gain theorem

Let H be a well-posed causal operator obtained by connecting in feedback two causal and weakly bounded operators H_1 and H_2. If

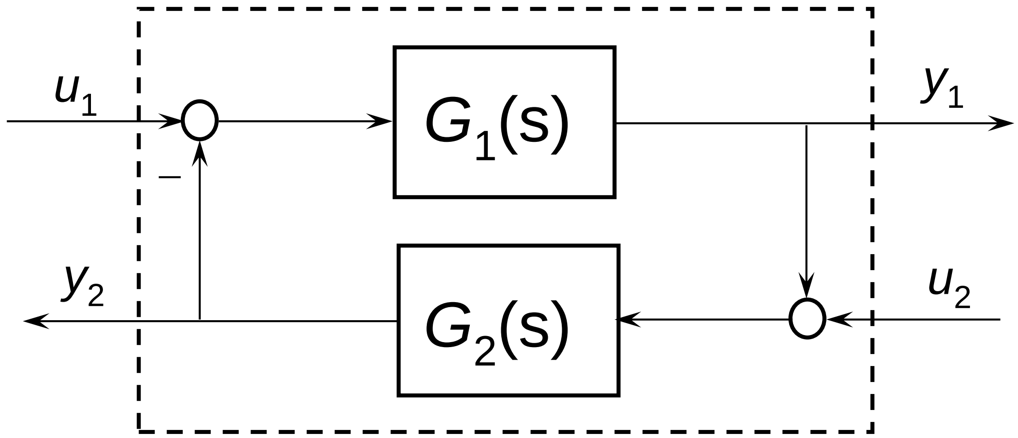
$$\lambda := \gamma(H_1)\gamma(H_2) < 1$$

then, H is weakly bounded. Furthermore,

$$\gamma(H_{11}) \leq \frac{\gamma(H_1)}{1-\lambda},\ \gamma(H_{22}) \leq \frac{\gamma(H_2)}{1-\lambda},\ \gamma(H_{12}),\gamma(H_{21}) \leq \frac{\lambda}{1-\lambda}$$

Remark: it holds irrespectively of the signs at the summation nodes.

STABILITY OF FEEDBACK LINEAR SYSTEMS



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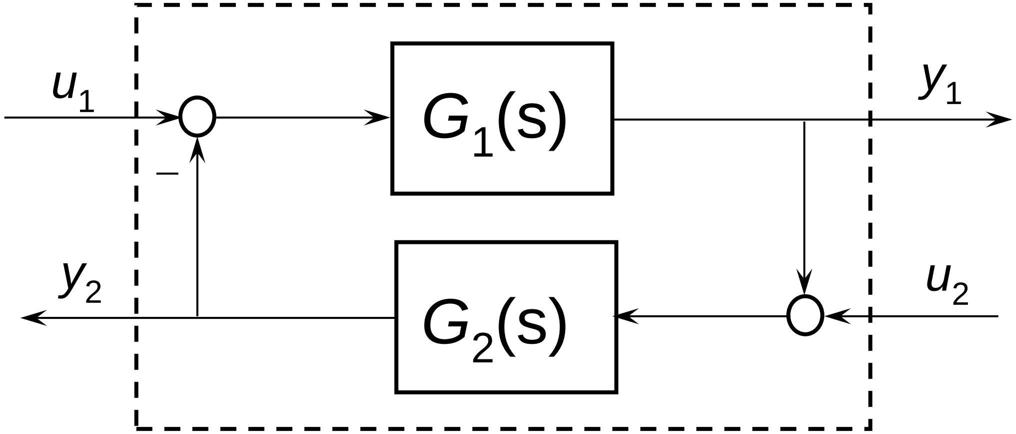

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In particular: if the two interconnected systems are asymptotically stable, then, the feedback system is asymptotically stable if

$$\sup_{\omega \in \Re^+} |G_1(j\omega)G_2(j\omega)| < 1$$

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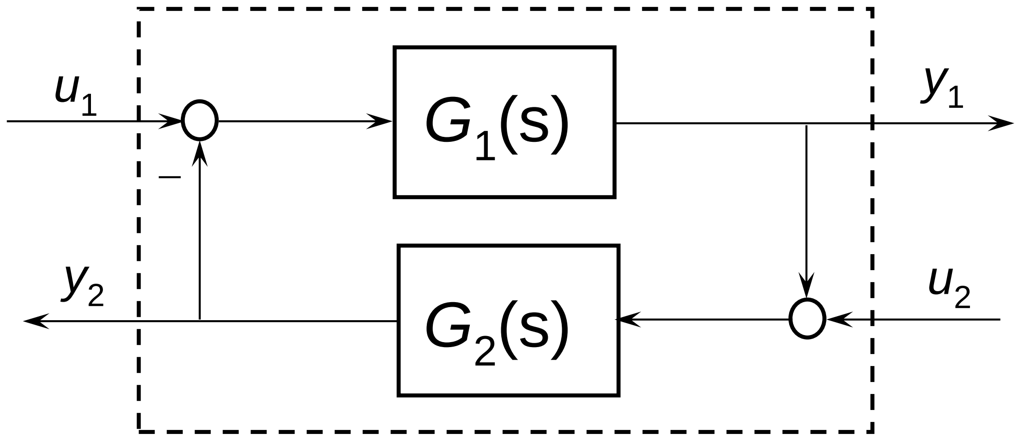

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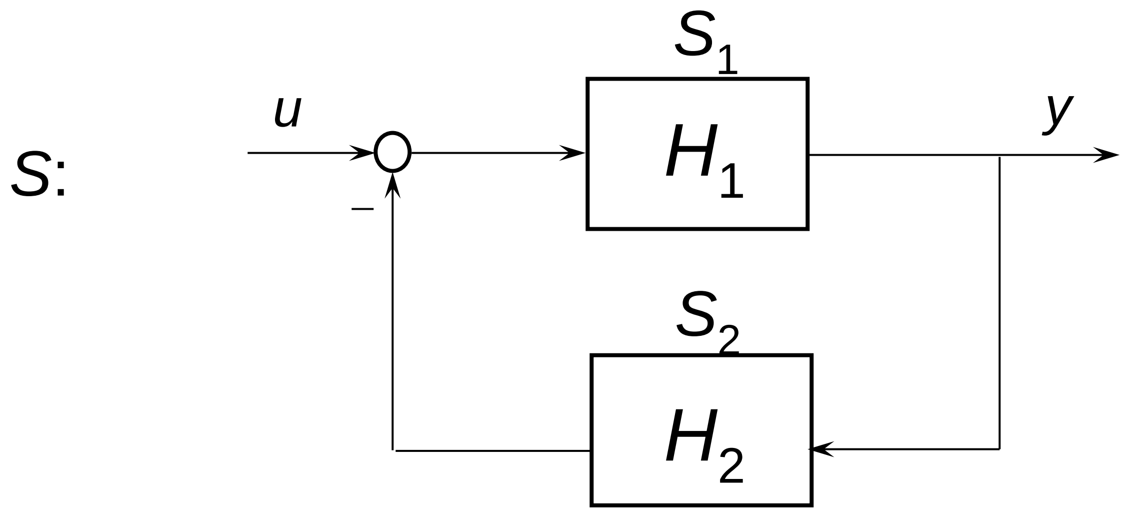
In turn, this condition is satisfied if

$$\Big(\sup_{\omega \in \Re^+} |G_1(j\omega)|\Big)\Big(\sup_{\omega \in \Re^+} |G_2(j\omega)|\Big) < 1$$

→ We have just shown a similar result for nonlinear systems.

EXAMPLE: LUR'E SYSTEM



S_1: linear time invariant dynamical system that is asymptotically stable and strictly proper with transfer function $G(s)$

→ causal and weakly bounded in L_p

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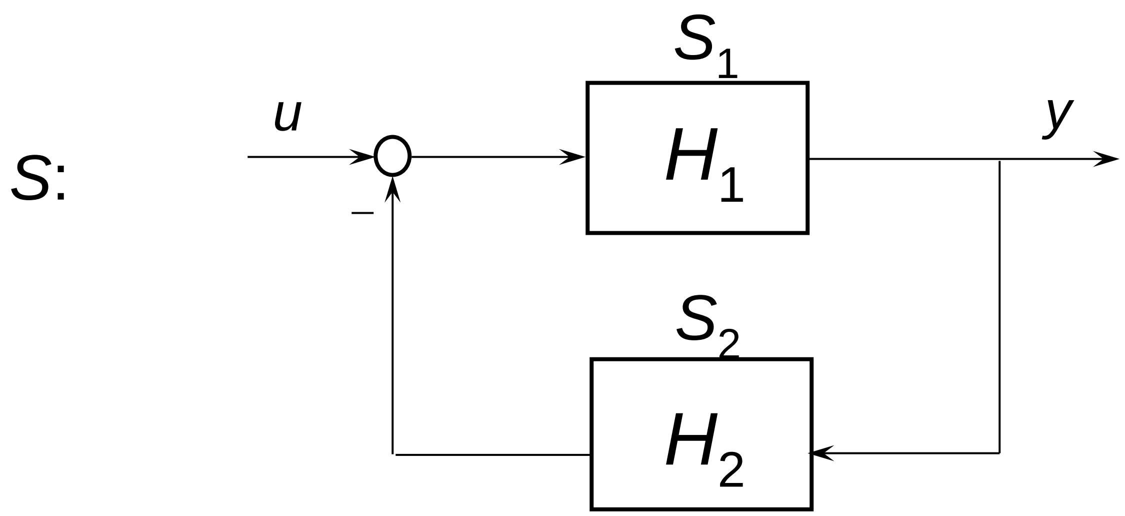

S_1: linear time invariant dynamical system that is asymptotically stable and strictly proper with transfer function *G(s)*

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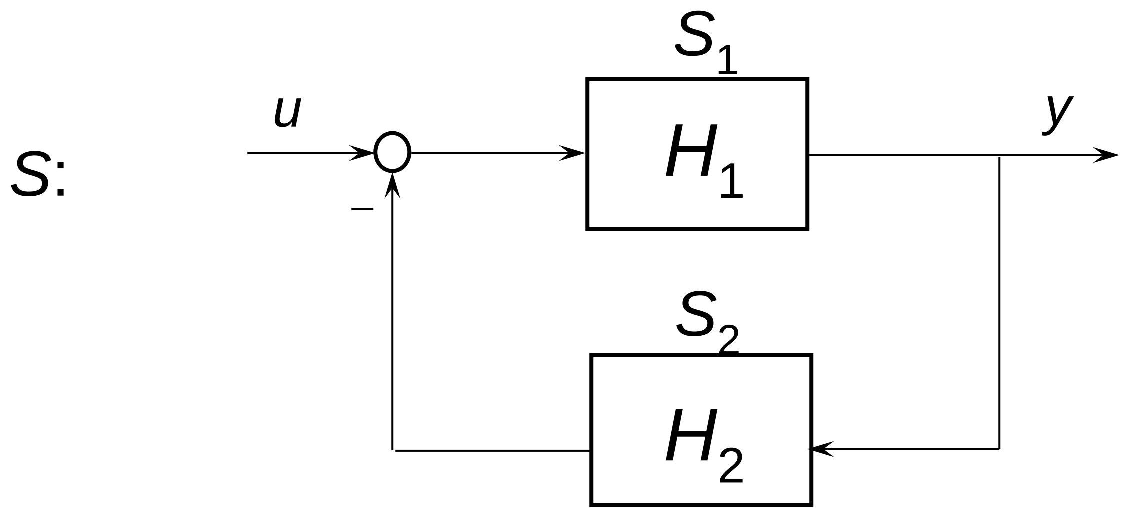
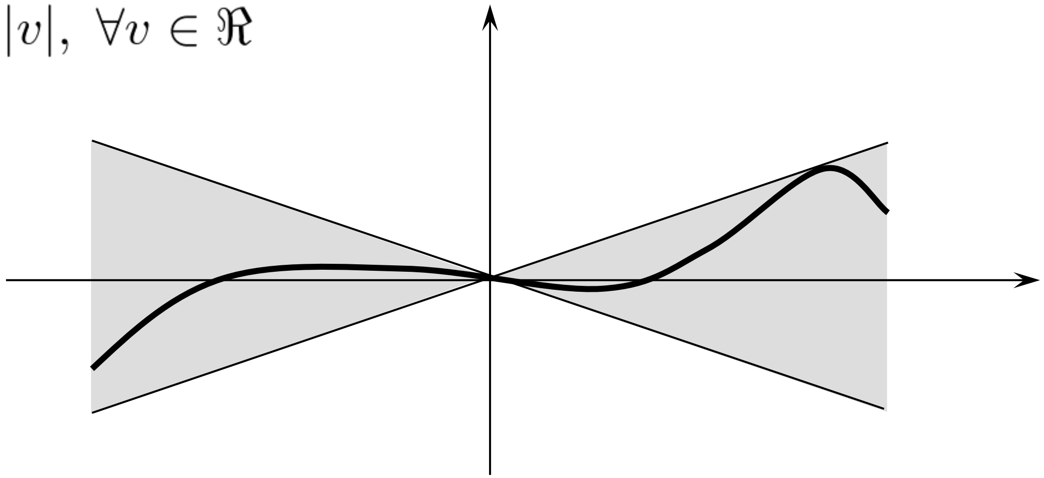
$$\gamma(H_1) = \gamma^\circ(G_1) = \begin{cases} \max\limits_{\omega \geq 0} |G(j\omega)| := G_{\max}, & \mathcal{L} = L_2 \\ \displaystyle\int_0^\infty |g(t)| dt := k_1, & \mathcal{L} = L_\infty \end{cases}$$

EXAMPLE: LUR'E SYSTEM



S_2: static system with sector nonlinearity $\varphi(\cdot)$ in [-*k*, *k*]

$$|\varphi(v)| \leq k|v|, \ \forall v \in \Re$$



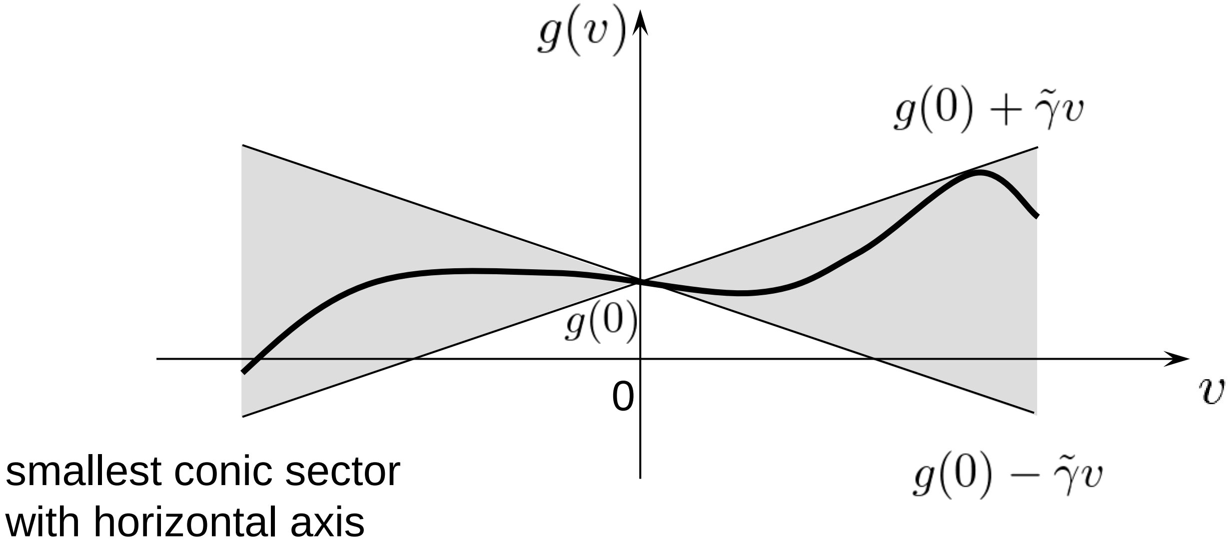
EXAMPLE 1: STATIC SYSTEM

$$S: \quad y(t) = g(u(t)), \forall t \in \Re^+$$

where $g : \Re \to \Re$ is piecewise continuous and $g(0) \neq 0$

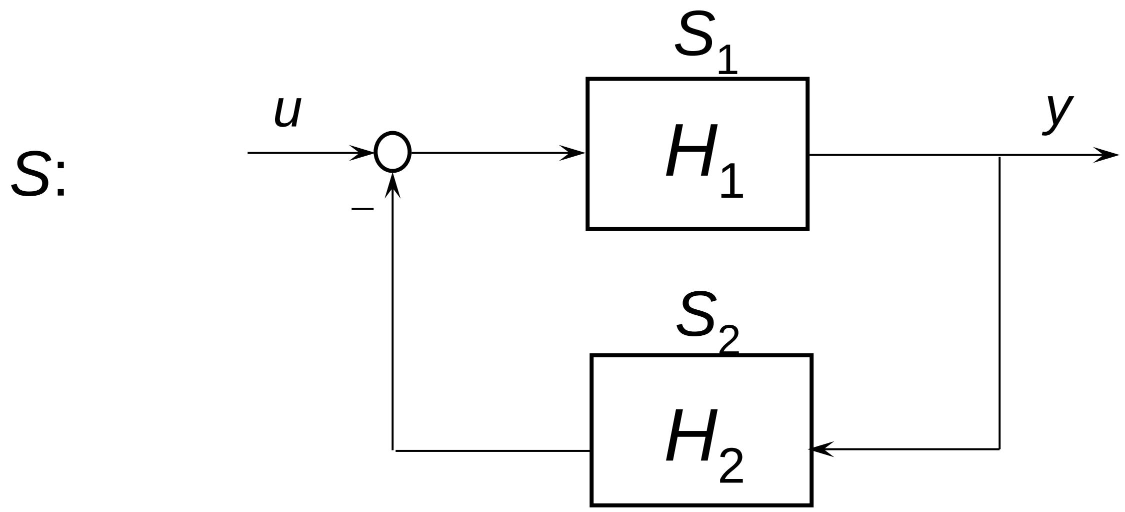
Set $\tilde{g}(v) := g(v) - g(0)$. Suppose that there exists some finite

$$\tilde{\gamma} := \inf\{k \in \Re^+ : \ |\tilde{g}(v)| \leq k|v|, \forall v \in \Re\}$$



Static system whose characteristic belongs to a conic sector

EXAMPLE: LUR'E SYSTEM



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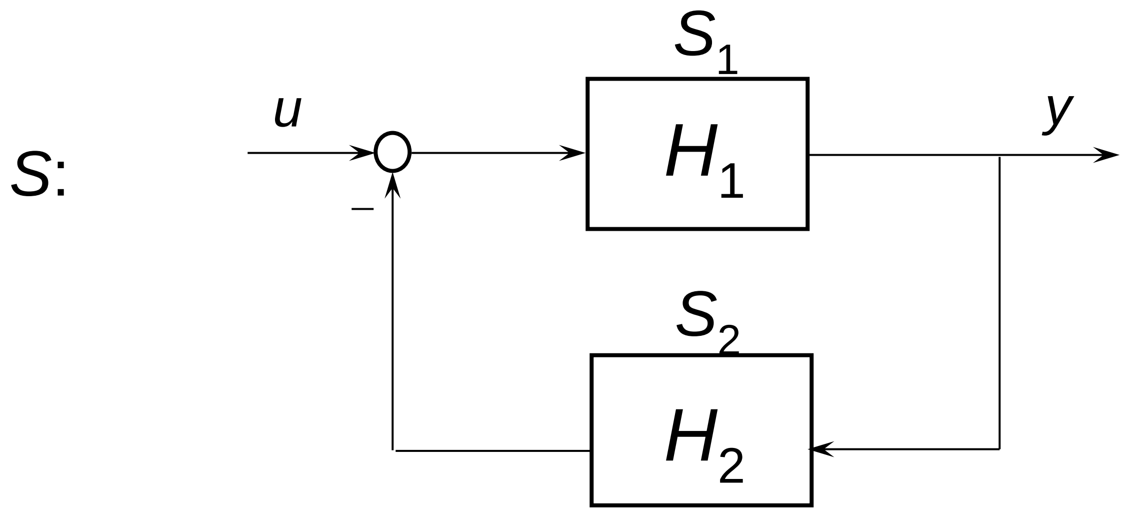

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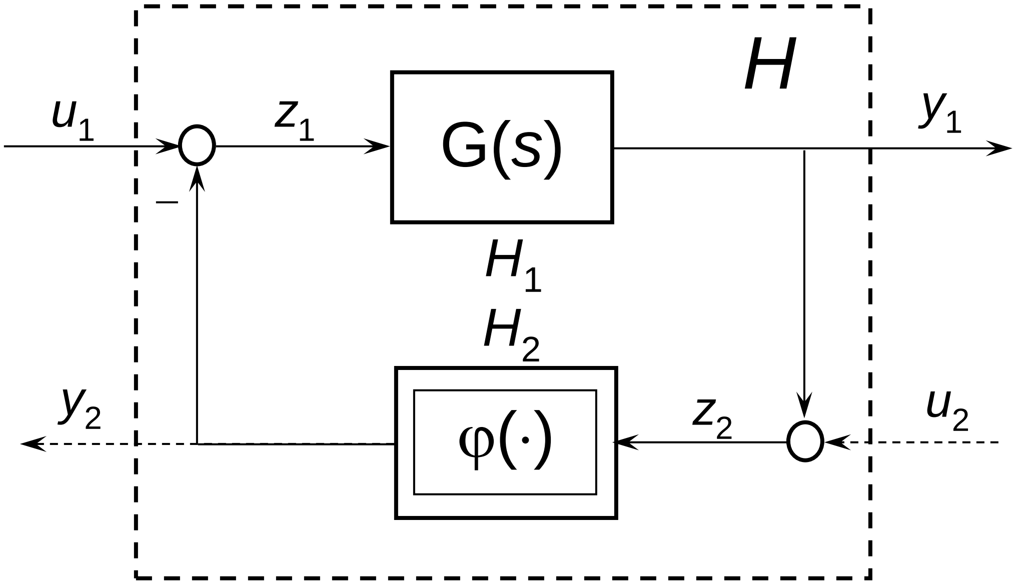
- $\mathcal{L} = L_2 \rightarrow \gamma(H_2) \le k$

  because

$$\|H_2(u(\cdot))\|_2^2 = \int_0^\infty \varphi^2(u(t))dt \le \int_0^\infty k^2 u^2(t)dt = k^2\|u(\cdot)\|_2^2, \ \forall u(\cdot) \in L_2$$

LUR'E SYSTEM: SMALL GAIN THEOREM

S :



System S (the associated operator H):

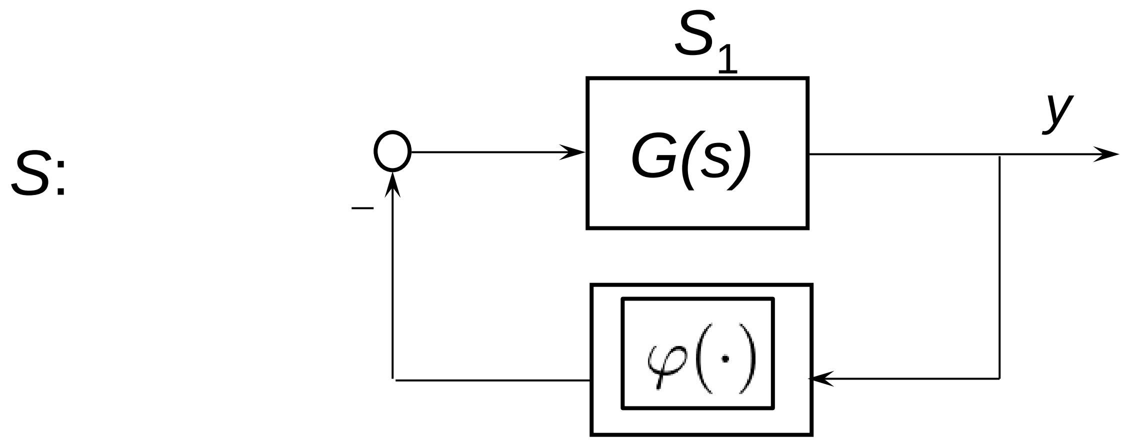
- is L_2-stable (for any sector nonlinearity $\varphi(\cdot)$ in [-*k*, *k*]) if

$$kG_{\max} < 1$$

- is L_∞-stable (for any sector nonlinearity $\varphi(\cdot)$ in [-*k*, *k*]) if

$$kk_1 < 1 \qquad (k_1 := \|g(\cdot)\|_1)$$

CIRCLE CRITERION (IN LYAPUNOV FORM)



Autonomous Lur'e system: absolute stability in sector [-*k*, *k*]

Necessary condition: S_1 asymptotically stable

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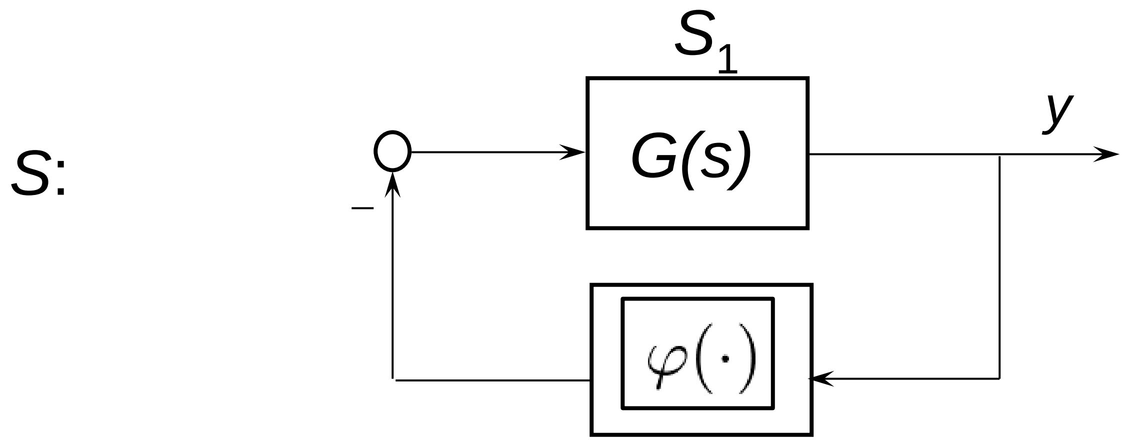

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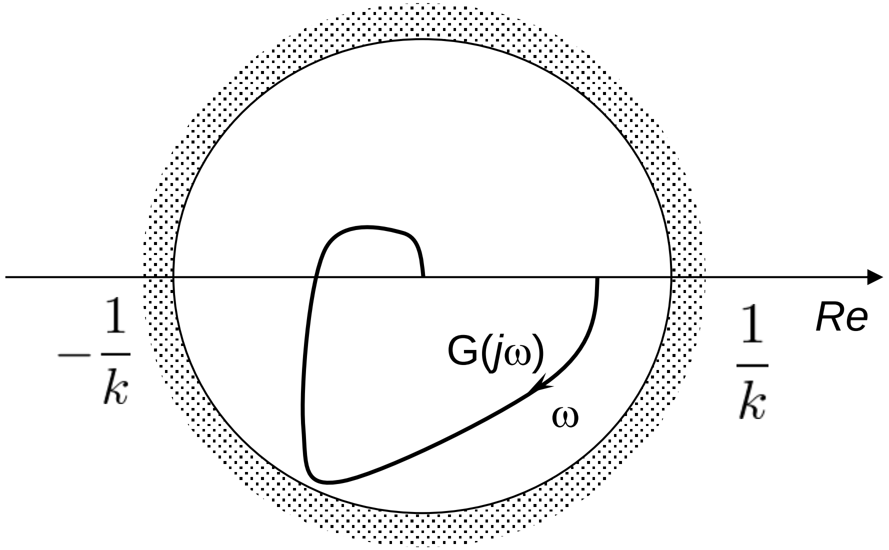

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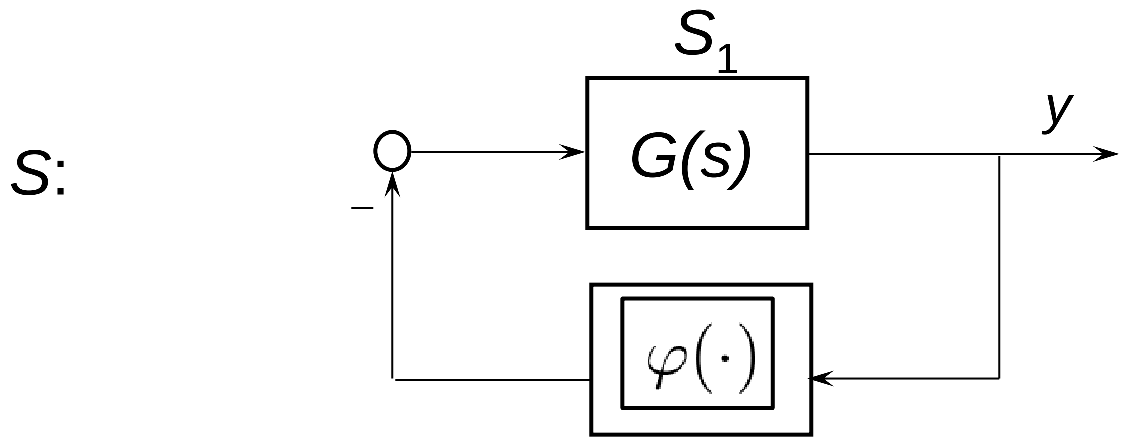

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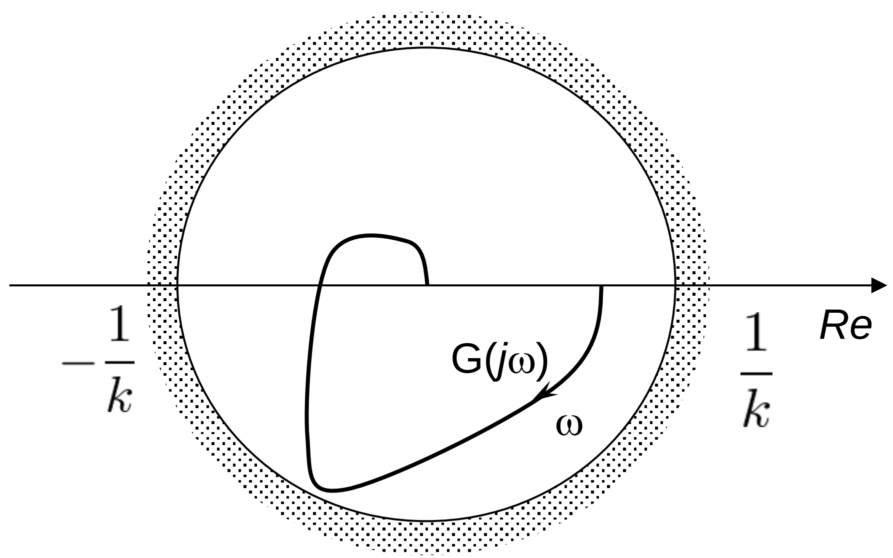
$$G_{\max} < \frac{1}{k} \Leftrightarrow kG_{\max} < 1$$

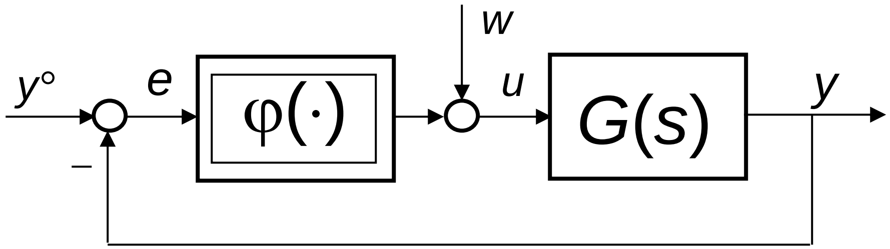


LUR'E SYSTEM: L_2 VERSUS ABSOLUTE STABILITY

- The connection between L_2-stability of a time-invariant Lur'e system and absolute stability of the same system with inputs sets to zero can be further strengthened by considering a generic sector $[k_1, k_2]$, $k_1 < k_2$ and formulating a Circle criterion for L_2 stability

L_2 STABILITY IN SECTOR $[k_1,k_2]$

S :



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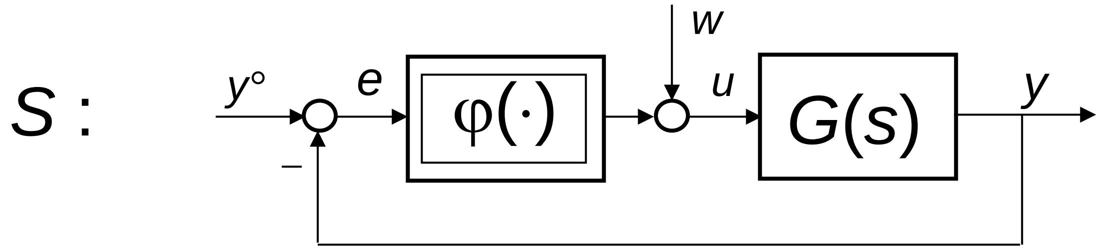

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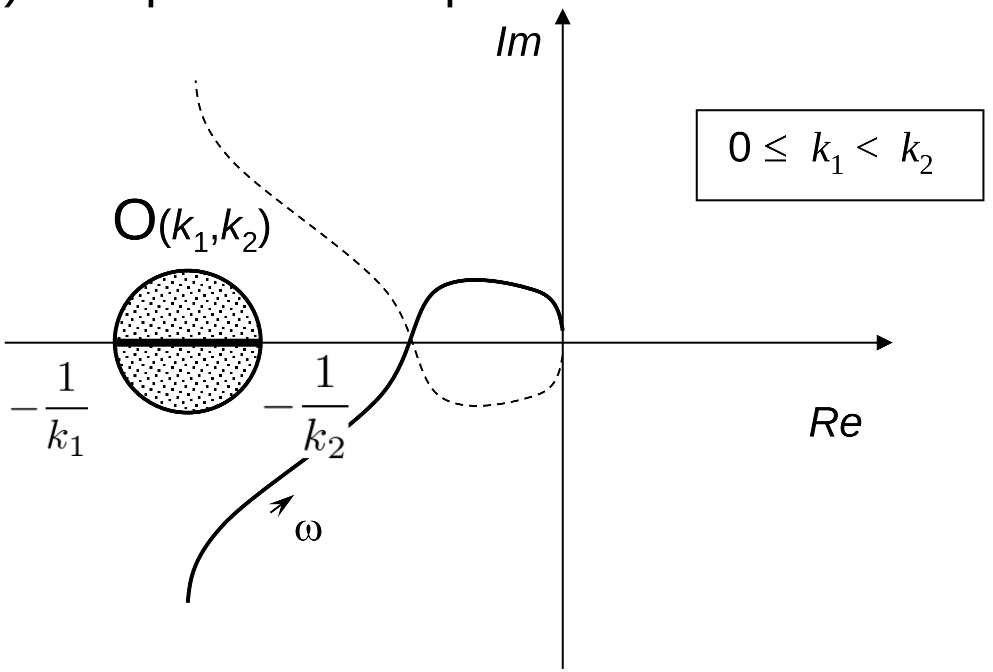
System S is L_2-stable for any $\varphi(\cdot) \in \Phi_{[k_1,k_2]}$ if the number of encirclements of G(s) Nyquist plot around $O(k_1,k_2)$ is equal to the number of poles of G(s) with positive real part.



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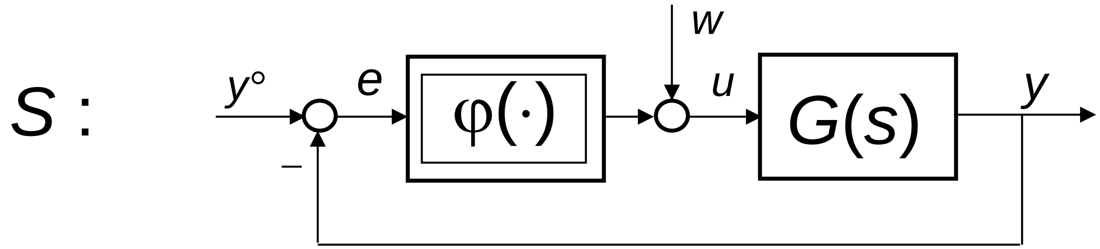

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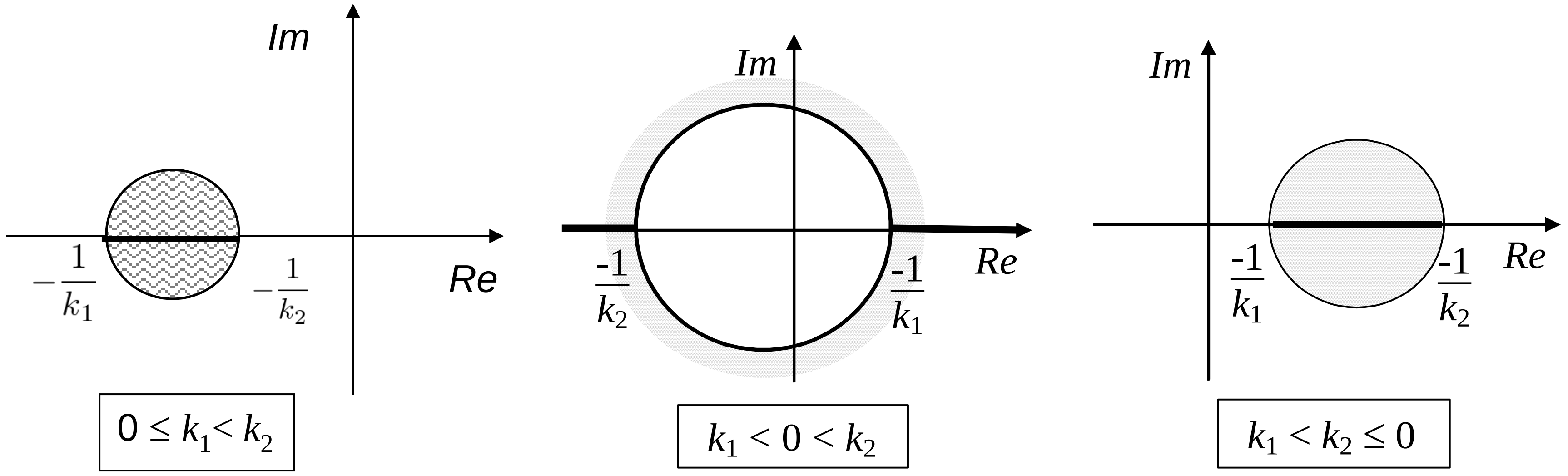

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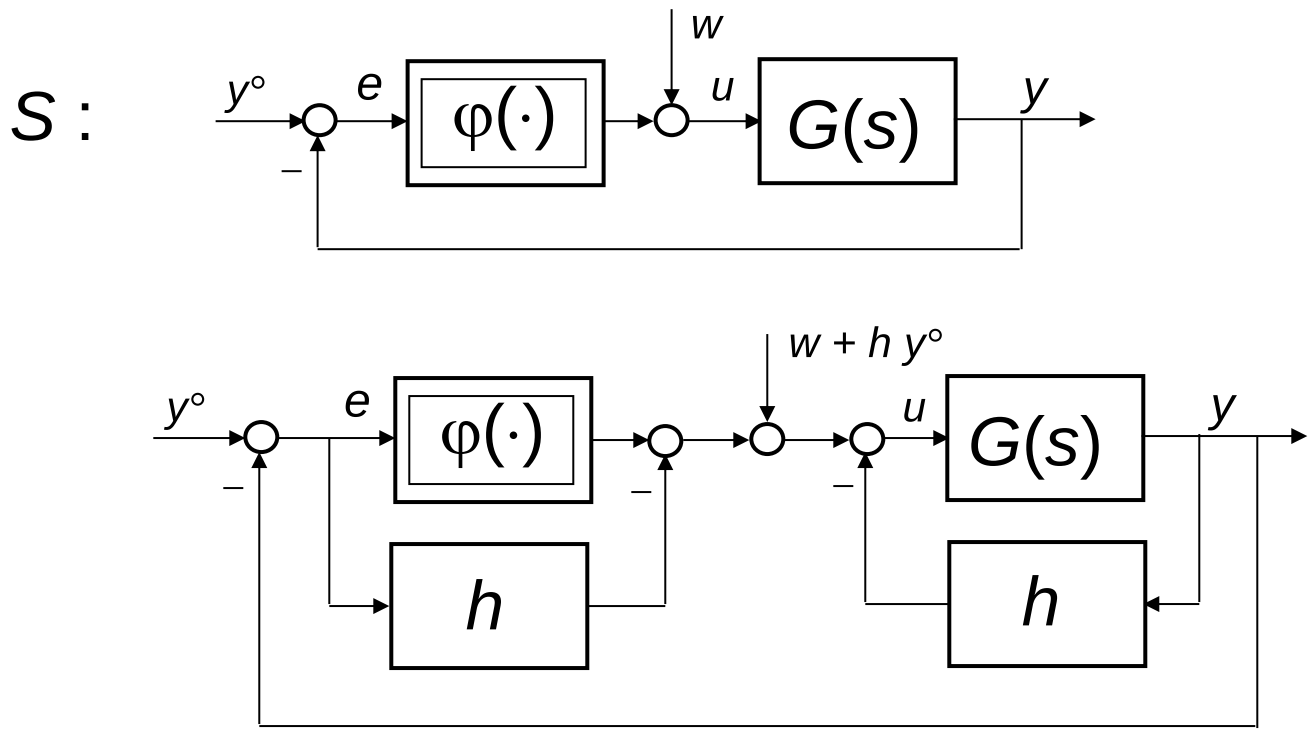

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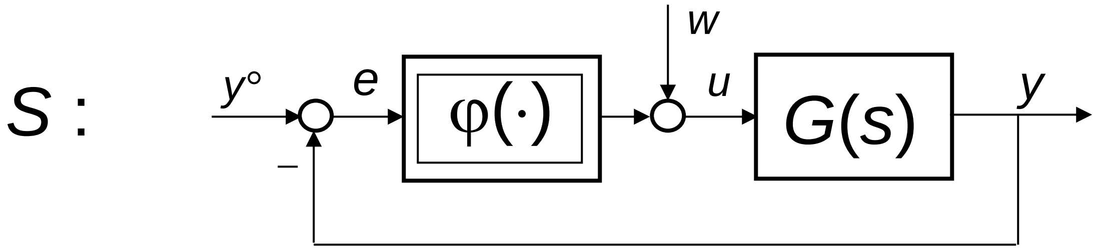

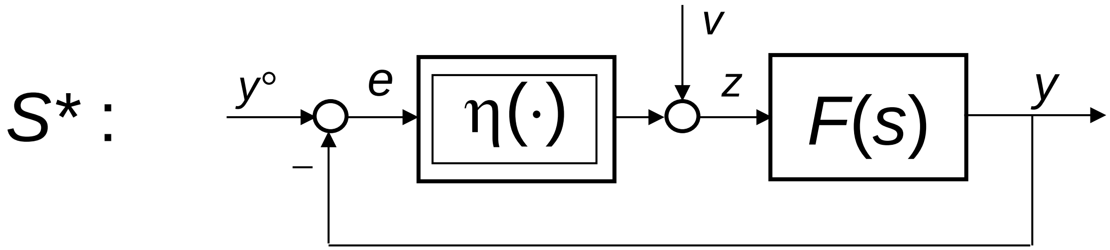

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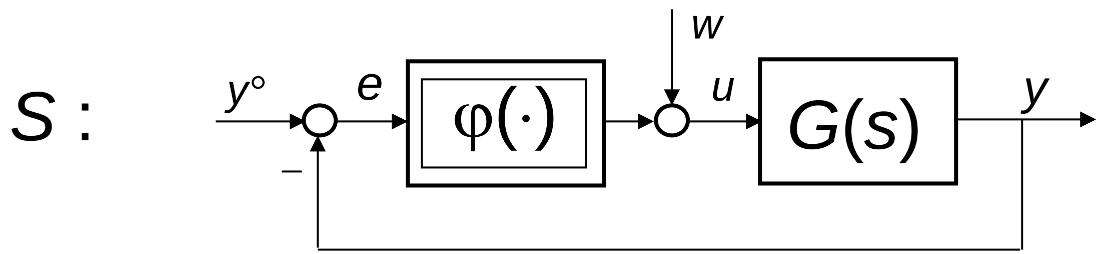

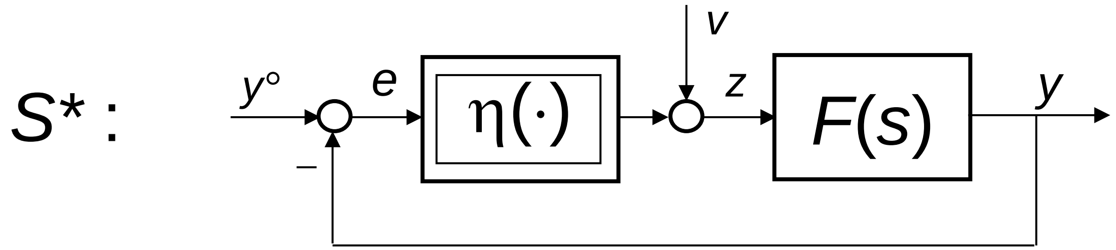

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$$h := \frac{k_1 + k_2}{2} \quad \Rightarrow \quad \varphi(\cdot) \in \Phi_{[k_1,k_2]} \Leftrightarrow \eta(\cdot) \in \Phi_{[-k,k]},\ k := \frac{k_2 - k_1}{2}$$

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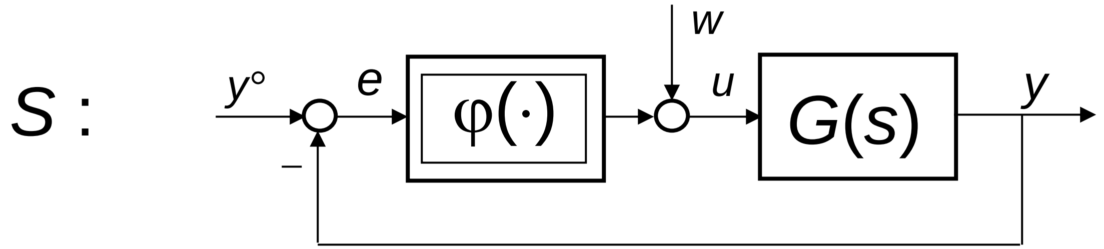

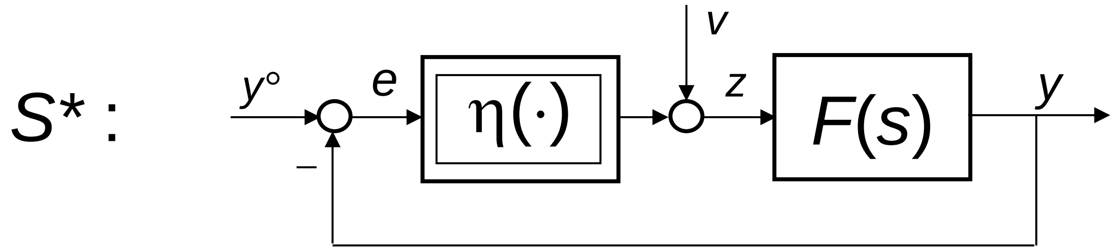

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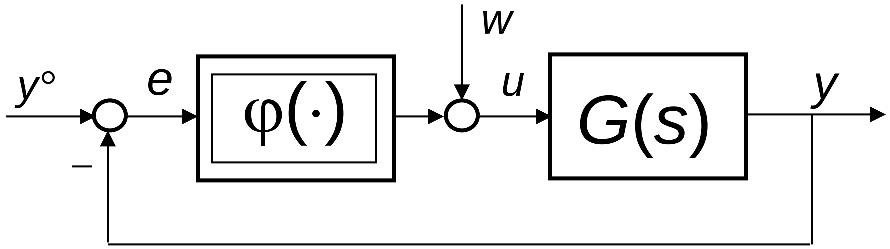

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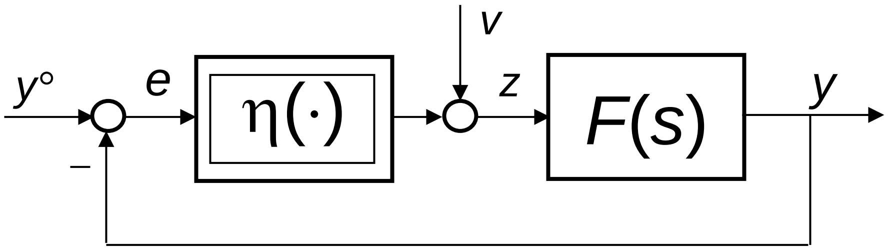

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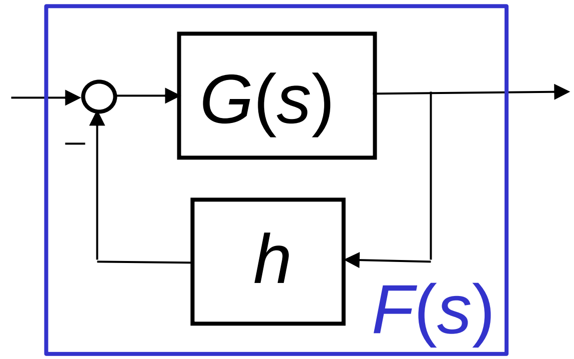
Remark: System S is L_2-stable in sector $[k_1, k_2]$ if and only if system S* is L_2-stable in sector $[-k, k]$

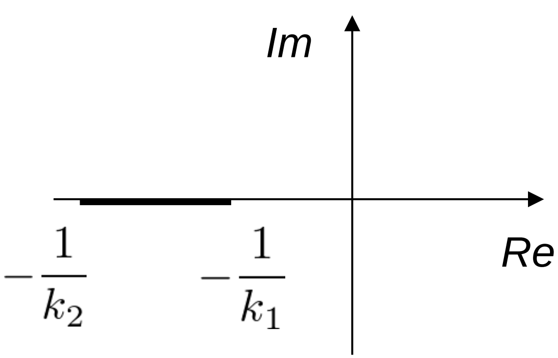
→ system with *F(s)* asymptotically stable and $F_{\max} < \dfrac{1}{k}$

L_2 STABILITY IN SECTOR $[k_1,k_2]$

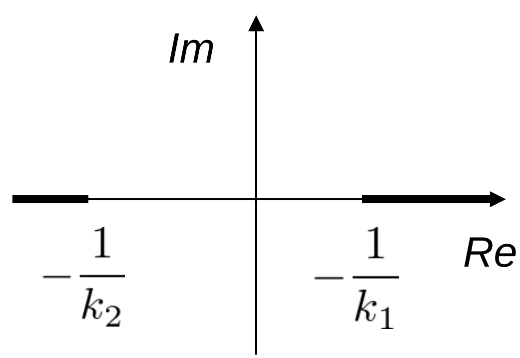


The poles of *F(s)* have negative real part since $h = \frac{k_1 + k_2}{2} \in [k_1, k_2]$ and G(s) Nyquist plot encircles $I(k_1,k_2)$ as many times as the number of poles of G(s) with positive real part

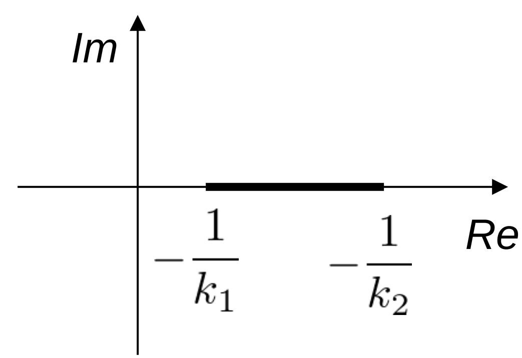
$0 \le k_1 < k_2$



$k_1 < 0 < k_2$



$k_1 < k_2 \le 0$

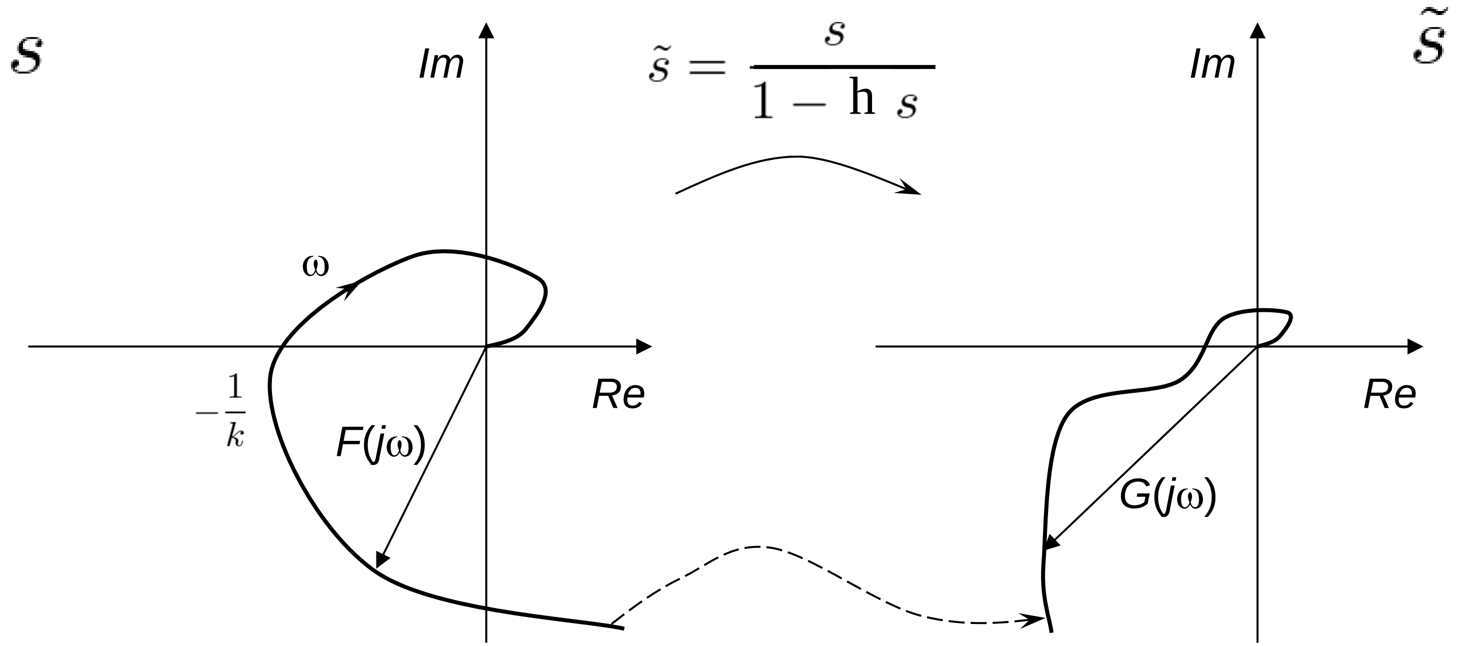


L_2 STABILITY IN SECTOR $[k_1,k_2]$

O(k_1, k_2) is the image through the mapping

$$F(s) \rightarrow \quad G(s) = \frac{F(s)}{1 - h\,F(s)}$$

of the region external to the circle of radius 1/k and center in the origin



L_2 STABILITY IN SECTOR $[k_1,k_2]$

$O(k_1, k_2)$ is the image through the mapping

$$F(s) \rightarrow \quad G(s) = \frac{F(s)}{1 - \mathrm{h}\, F(s)}$$

of the region external to the circle of radius 1/k and center in the origin

Then, if G(s) Nyquist plot does not inteserct $O(k_1, k_2)$, F(s) Nyquist plot is within that circle, i.e.,

$$F_{\max} < \frac{1}{k}$$

which concludes the proof.