NUMERICAL EXAMPLES:
FEEDBACK LINEARIZATION

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) ≠ 0, and fully measurable state:

1. Provide a necessary and sufficient condition for the system to be fully linearizable in x°=0 via static state feedback.

STATE FEEDBACK FULL LINEARIZABILITY

Theorem:
There exists a solution to the problem of state feedback full linearizability in *x°* if and only if one can find a regular function *c(·)* such that system

$$S: \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

has relative degree *r* equal to the order *n* of the system in x°=0.

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o: \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) ≠ 0, and fully measurable state:

1. Provide a necessary and sufficient condition for the system to be fully linearizable in x°=0 via static state feedback.

2. Check that such a condition is satisfied for the second order system with

$$a(x) = \begin{bmatrix} x_2^3 - x_1 \\ -x_1 \end{bmatrix} \qquad b(x) = \begin{bmatrix} x_1^2 + 1 \\ 0 \end{bmatrix}$$

(suggestion: try with $y=x_1$ and $y=x_2$)

RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition

The relative degree *r* of a system *S* in *x*° is given by the minimum order of the time derivative of the output *y* that is affected directly by the input *u* when computed in x°

$$S: \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o: \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) $\neq$ 0, and fully measurable state:

2. Check that such a condition is satisfied for the second order system with

$$a(x) = \begin{bmatrix} x_2^3 - x_1 \\ -x_1 \end{bmatrix} \qquad b(x) = \begin{bmatrix} x_1^2 + 1 \\ 0 \end{bmatrix}$$

(suggestion: try with $y=x_1$ and $y=x_2$)

Solution

$$y = x_1 \rightarrow \dot{y} = x_2^3 - x_1 + (x_1^2+1)u$$
$$(x_1^2+1) = 1 \text{ in } x = 0 \ \rightarrow \mathrm{r} = 1 < \mathrm{n} = 2$$

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) ≠ 0, and fully measurable state:

2. Check that such a condition is satisfied for the second order system with

$$a(x) = \begin{bmatrix} x_2^3 - x_1 \\ -x_1 \end{bmatrix} \qquad b(x) = \begin{bmatrix} x_1^2 + 1 \\ 0 \end{bmatrix}$$

(suggestion: try with $y=x_1$ and $y=x_2$)

<u>Solution</u>

$$y = x_2 \rightarrow \dot{y} = -x_1 \rightarrow \ddot{y} = -x_2^3 + x_1 - (x_1^2+1)u$$

$$(x_1^2+1) = 1 \text{ in } x = 0 \ \rightarrow \mathrm{r} = 2 = \mathrm{n}$$

→ Fully linearizable

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o: \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) ≠ 0, and fully measurable state:

1. Provide a necessary and sufficient condition for the system to be fully linearizable in x°=0 via static state feedback.

2. Check that such a condition is satisfied for the second order system with

$$a(x) = \begin{bmatrix} x_2^3 - x_1 \\ -x_1 \end{bmatrix} \qquad b(x) = \begin{bmatrix} x_1^2 + 1 \\ 0 \end{bmatrix}$$

(suggestion: try with $y=x_1$ and $y=x_2$)

3. derive the static state feedback control law that linearizes the system in x°=0 and assigns the poles to be {-1, -10}.

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) $\neq$ 0, and fully measurable state:

3. derive the static state feedback control law that linearizes the system in x°=0 and assigns the poles to be {-1, -10}.

<u>Solution</u>

$$\ddot{y} = -x_2^3 + x_1 - (x_1^2+1)u$$

If we set u such that

$$-x_2^3 + x_1 - (x_1^2+1)u = -k_1\dot{y} - k_2 y + v$$

Then, we get the linear closed loop system

$$\ddot{y} + k_1\,\dot{y} + k_2 y = v$$

and if $k_1 = 11$ and $k_2 = 10$, poles will be {-1, -10}.

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) ≠ 0, and fully measurable state:

3. derive the static state feedback control law that linearizes the system in x°=0 and assigns the poles to be {-1, -10}.

Solution

The static state feedback control law is then obtained from

$$-x_2^3 + x_1 - (x_1^2+1)u = -11\dot{y} - 10y + v$$

and since $y = x_2$ and $\dot{y} = -x_1$, it is given by

$$u = \frac{1}{x_1^2 + 1}(-x_2^3+x_1 - 11x_1 + 10x_2 - v)$$

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) ≠ 0, and fully measurable state:

1. Provide a necessary and sufficient condition for the system to be fully linearizable in x°=0 via static state feedback.

2. Check that such a condition is satisfied when

$$a(x) = \begin{bmatrix} x_2^3 - x_1 \\ -x_1 \end{bmatrix} \qquad b(x) = \begin{bmatrix} x_1^2 + 1 \\ 0 \end{bmatrix}$$

(suggestion: try with $y=x_1$ and $y=x_2$)

4. Set $y=x_2$ and write the system in normal canonical form.

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) ≠ 0, and fully measurable state:

1. Provide a necessary and sufficient condition for the system to be fully linearizable in x°=0 via static state feedback.

2. Check that such a condition is satisfied when

$$a(x) = \begin{bmatrix} x_2^3 - x_1 \\ -x_1 \end{bmatrix} \qquad b(x) = \begin{bmatrix} x_1^2 + 1 \\ 0 \end{bmatrix}$$

(suggestion: try with $y=x_1$ and $y=x_2$)

4. Set $y=x_2$ and write the system in normal canonical form.

<u>Solution:</u> relative degree is r=n=2 then

$\tilde{x}_1 = y = x_2$ and $\tilde{x}_2 = \dot{y} = -x_1$

EXAMPLE 1

Given the nonlinear regular SISO system governed by the

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) ≠ 0, and fully measurable state:

1. Provide a necessary and sufficient condition for the system to be fully linearizable in x°=0 via static state feedback.

2. Check that such a condition is satisfied when

$$a(x) = \begin{bmatrix} x_2^3 - x_1 \\ -x_1 \end{bmatrix} \qquad b(x) = \begin{bmatrix} x_1^2 + 1 \\ 0 \end{bmatrix}$$

(suggestion: try with $y=x_1$ and $y=x_2$)

4. Set $y=x_2$ and write the system in normal canonical form.

<u>Solution:</u> $\tilde{x}_1 = y = x_2$ and $\tilde{x}_2 = \dot{y} = -x_1$

$$\tilde{a}(\tilde{x}) = \begin{bmatrix} \tilde{x}_2 \\ -\tilde{x}_1^3 - \tilde{x}_2 \end{bmatrix} \qquad \tilde{b}(\tilde{x}) = \begin{bmatrix} 0 \\ -\tilde{x}_2^2 - 1 \end{bmatrix}$$

EXAMPLE 2

Given the nonlinear regular SISO system governed by the

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with a(0) = 0, b(0) $\neq$ 0, and fully measurable state, we know that there exists a solution to the problem of state feedback full linearizability in *x°* <u>if and only if</u> one can find a regular function *c(·)* such that system

$$S : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

has relative degree *r* equal to the order *n* of the system in x°=0.

Provide an example where this condition is satisfied and another example where it is not.

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with

$$\mathrm{a(x)} = \begin{bmatrix} -\mathrm{x}_2 \\ -\mathrm{x}_2^3 \end{bmatrix} \qquad \mathrm{b(x)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y = x_1 \to \dot{y} = -x_2 \to \ddot{y} = x_2^3 - u \to \mathrm{r} = 2 = \mathrm{n}$$

there exists a solution to the problem of state feedback full linearizability

STATE FEEDBACK FULL LINEARIZABILITY PROBLEM

Practical impact of this theorem?

There exists a solution to the problem of state feedback full linearizability in $x°= 0$ if and only if one can find a regular function $c(\cdot)$ such that system

$$S: \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

has relative degree *r* equal to the order *n* of the system in x°=0, i.e.

$$\begin{cases} L_b c = 0 \\ L_b L_a c = 0 \\ \vdots \\ L_b L_a^{n-2} c = 0 \end{cases}$$ in a neighborhood of x°=0

with $[L_b L_a^{n-1} c]_{x^\circ} \neq 0$

→ Find a solution to *n* - 1 nonlinear partial differential equations…

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with

$$a(x) = \begin{bmatrix} 0 \\ x_2^3 \end{bmatrix} \qquad b(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_2^3 \end{cases}$$

there exists no solution to the problem of state feedback full linearizability since we cannot get access to the (nonlinear) dynamics of the second state variable

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

with

$$a(x) = \begin{bmatrix} 0 \\ x_2^3 \end{bmatrix} \qquad b(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If $y = c(x_1, x_2)$ then

$$L_b c(x_1, x_2) = \begin{bmatrix} \frac{\partial c}{\partial x_1} & \frac{\partial c}{\partial x_2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\partial c}{\partial x_1}(x_1, x_2)$$

$\rightarrow \frac{\partial c}{\partial x_1}(x_1, x_2) = 0$ in a neighborhood of $(x_1, x_2) = (0,0)$

$$L_b L_a c(x_1, x_2) = L_b \begin{bmatrix} \frac{\partial c}{\partial x_1} & \frac{\partial c}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ x_2^3 \end{bmatrix} = \frac{\partial}{\partial x} \begin{bmatrix} \frac{\partial c}{\partial x_2} x_2^3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\rightarrow \frac{\partial^2 c}{\partial x_1 \partial x_2} x_2^3 \neq 0$ in $(x_1, x_2) = (0,0)$ not possible

EXERCISE 5 OF THE EXAM DATED JULY 20, 2017

Consider the SISO nonlinear regular system of order n, described by

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

1. state the necessary and sufficient condition for (local) state feedback full linearization in a given state $x^o \epsilon R^n$

STATE FEEDBACK FULL LINEARIZABILITY

Theorem:
There exists a solution to the problem of state feedback full linearizability in $x°$ <u>if and only if</u> one can find a regular function *c(·)* such that system

$$S: \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

has relative degree *r* equal to the order *n* of the system in x°.

EXERCISE 5 OF THE EXAM DATED JULY 20, 2017

Consider the SISO nonlinear regular system of order n, described by

$$S_o: \quad \dot{x} = a(x) + b(x)u$$

2. determine if the system of order n=2 with

$$a(x) = \begin{bmatrix} 2\sin(x_2) + x_1 - 1 \\ -x_1^2 \end{bmatrix} \qquad b(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is locally fully linearizable by state feedback in $x^o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If that is the case, write the state feedback control law that makes the feedback system linear with eigenvalues all identical to -1.

EXERCISE 5 OF THE EXAM DATED JULY 20, 2017

Consider the SISO nonlinear regular system of order n, described by

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

2. determine if the system of order n=2 with

$$a(x) = \begin{bmatrix} 2\sin(x_2) + x_1 - 1 \\ -x_1^2 \end{bmatrix} \qquad b(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is locally fully linearizable by state feedback in $x^o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

RELATIVE DEGREE OF A NONLINEAR SYSTEM

Definition

The relative degree *r* of a system *S* in *x*° is given by the minimum order of the time derivative of the output *y* that is affected directly by the input *u* when computed in x°

$$S: \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) \end{cases}$$

EXERCISE 5 OF THE EXAM DATED JULY 20, 2017

Consider the SISO nonlinear regular system of order n, described by

$$S_o: \quad \dot{x} = a(x) + b(x)u$$

2. determine if the system of order n=2 with

$$a(x) = \begin{bmatrix} 2\sin(x_2) + x_1 - 1 \\ -x_1^2 \end{bmatrix} \qquad b(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is locally fully linearizable by state feedback in $x^o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

<u>Solution</u> $y = x_1$

$\rightarrow \ \dot{y} = 2\sin(x_2) + x_1 - 1$

$\rightarrow \ddot{y} = 2\cos(x_2)(-x_1^2 + u) + 2\sin(x_2) + x_1 - 1$

since $[2\cos(x_2)]|_{x^o} = 2\cos(1) \neq 0$, then, $r = 2 = n$ hence the system is locally fully linearizable in x^o

EXERCISE 5 OF THE EXAM DATED JULY 20, 2017

Consider the SISO nonlinear regular system of order n, described by

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

2. determine if the system of order n=2 with

$$a(x) = \begin{bmatrix} 2\sin(x_2) + x_1 - 1 \\ -x_1^2 \end{bmatrix} \qquad b(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is locally fully linearizable by state feedback in $x^o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If that is the case, write the state feedback control law that makes the feedback system linear with eigenvalues all identical to -1.

EXERCISE 5 OF THE EXAM DATED JULY 20, 2017

Consider the SISO nonlinear regular system of order n, described by

$$S_o: \quad \dot{x} = a(x) + b(x)u$$

If that is the case, write the state feedback control law that makes the feedback system linear with eigenvalues all identical to -1.

Solution

$$\ddot{y} = 2\cos(x_2)(-x_1^2 + u) + 2\sin(x_2) + x_1 - 1$$

If we set u such that

$$2\cos(x_2)(-x_1^2 + u) + 2\sin(x_2) + x_1 - 1 = -k_1\dot{y} - k_2 y + v$$

Then, we get the linear closed loop system

$$\ddot{y} + k_1\dot{y} + k_2 y = v$$

and if $k_1 = 2$, $k_2 = 1$, poles (hence eigenvalues) will be {-1, -1}.

EXERCISE 5 OF THE EXAM DATED JULY 20, 2017

Consider the SISO nonlinear regular system of order n, described by

$$S_o : \quad \dot{x} = a(x) + b(x)u$$

If that is the case, write the state feedback control law that makes the feedback system linear with eigenvalues all identical to -1.

Solution

The requested state feedback control law is then obtained from

$$2\cos(x_2)(-x_1^2 + u) + 2\sin(x_2) + x_1 - 1 = -2\dot{y} - y + v$$

and it is given by

$$u = x_1^2 + \frac{1}{2\cos(x_2)}(-2\sin(x_2) - x_1 + 1 - 2(2\sin(x_2) + x_1 - 1) - x_1 + v)$$