Lyapunov stability

ORDINARY DIFFERENTIAL EQUATIONS

An ordinary differential equation is a mathematical model of a continuous state continuous time system:

$$\dot{x}(t) = f(x(t))$$

$X = \Re^n$	$\equiv$ state space
$f: \Re^n \to \Re^n$	$\equiv$ vector field (assigns a "velocity" vector to each x)

Given an initial value $x_0 \in X$,
an execution (solution in the sense of Caratheodory) over the time interval [0,T) is a function $x: [0,T) \to \Re^n$ such that:

- $x(0) = x_0$
- x is continuous and piecewise differentiable
- $x(t) = x(0) + \int_0^t f(x(\tau))d\tau, \ \forall t \in [0,T)$

ODE SOLUTION: WELL-POSEDNESS

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0$$

Theorem [global existence and uniqueness]

If f: $\Re^n \to \Re^n$ is globally Lipschitz continuous, then $\forall$ x_0 there exists a single solution with $x(0)=x_0$ defined on $[0,\infty)$.

Def. f: $\Re^n \to \Re^n$ is globally Lipschitz continuous, if there exists a constant L such that

$\|f(x_1)-f(x_2)\| \leq L \|x_1-x_2\|, \ \forall\ x_1,x_2 \in \Re^n$

STABILITY OF CONTINUOUS SYSTEMS

$$\dot{x}(t) = f(x(t))$$

with f: $\Re^n \to \Re^n$ globally Lipschitz continuous

Definition (equilibrium):

$x_e \in \Re^n$ for which $f(x_e)=0$

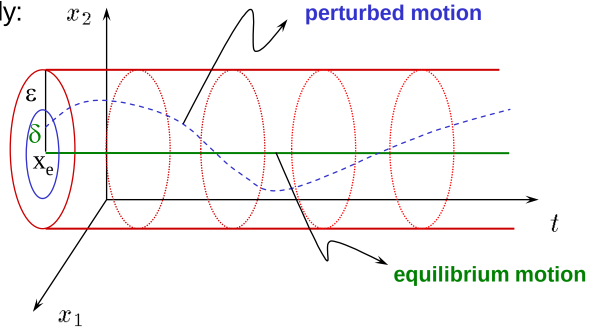
Definition (stable equilibrium):

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq 0$$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

execution starting from $x(0)=x_0$

Graphically:



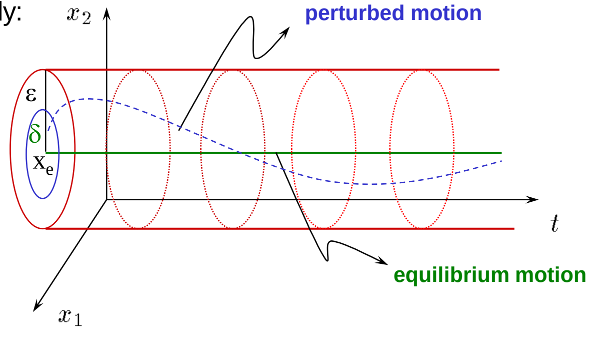
small perturbations lead to small changes in behavior

Definition (asymptotically stable equilibrium):

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| < \delta \rightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq 0$$

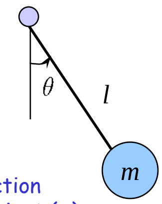
and δ can be chosen so that $\lim_{t \to \infty} (x(t) - x_e) = 0$

Graphically:



small perturbations lead to small changes in behavior and are re-absorbed, in the long run

EXAMPLE: PENDULUM

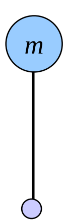


friction coefficient (α)

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{\alpha}{m}x_2$$

EXAMPLE: PENDULUM



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$$x_e = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$ unstable equilibrium

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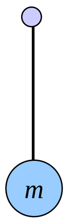

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$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ as. stable equilibrium

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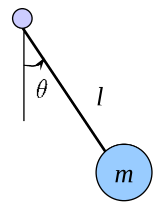

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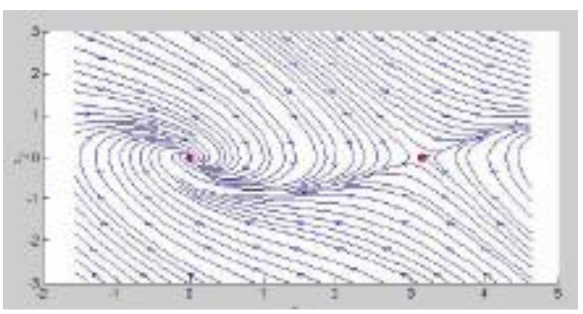


Let x_e be asymptotically stable.

Definition (domain of attraction):

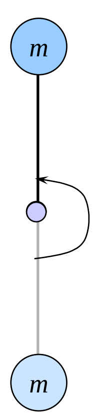
The domain of attraction of x_e is the set of x_0 such that

$$\lim_{t \to \infty} (x(t) - x_e) = 0$$

execution starting from $x(0)=x_0$

Definition (globally asymptotically stable equilibrium):

x_e is globally asymptotically stable (GAS) if its domain of attraction is the whole state space $\Re^n$



EXAMPLE: PENDULUM

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{\alpha}{m}x_2$$

$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

as. stable equilibrium

small perturbations are absorbed, not all perturbations → not GAS

Let x_e be asymptotically stable.

Definition (exponential stability):

x_e is exponentially stable if $\exists\ \alpha, \delta, \beta > 0$ such that

$$\|x_0 - x_e\| < \delta \rightarrow \|x(t) - x_e\| \leq \alpha \|x_0 - x_e\| e^{-\beta t}, \forall t \geq 0$$

STABILITY OF CONTINUOUS SYSTEMS

$$\dot{x}(t) = f(x(t))$$

with f: $\Re^n \rightarrow \Re^n$ globally Lipschitz continuous

Definition (equilibrium):

$x_e \in \Re^n$ for which $f(x_e)=0$

Without loss of generality we suppose that

$x_e = 0$

if not, then $z := x - x_e \rightarrow$ $dz/dt = g(z)$, $g(z) := f(z+x_e)$ $(g(0) = 0)$

STABILITY OF CONTINUOUS SYSTEMS

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0$$

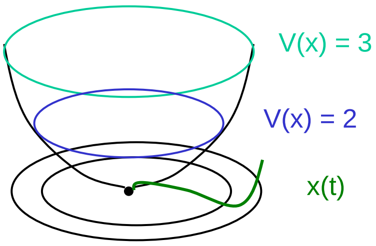
with f: $\Re^n \to \Re^n$ globally Lipschitz continuous

How to prove stability?

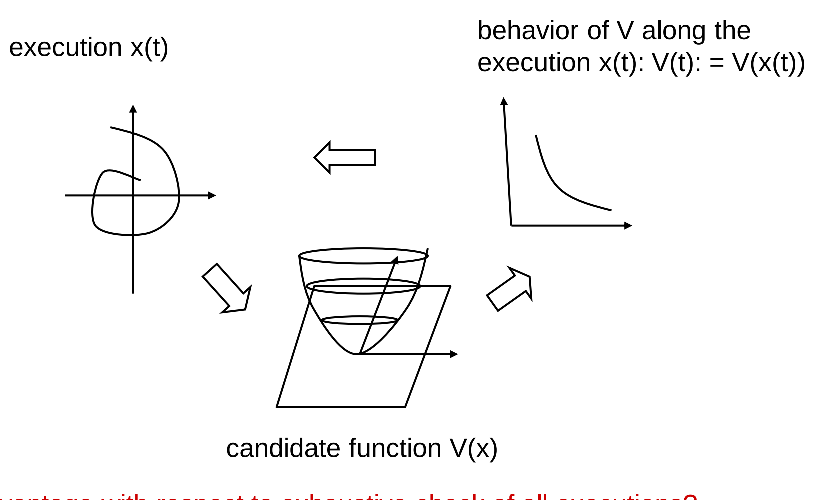
find a function V: $\Re^n \to \Re$ such that

V(0) = 0 and V(x) >0, for all x $\neq$ 0

V(x) is decreasing along the executions of the system



STABILITY OF CONTINUOUS SYSTEMS



Advantage with respect to exhaustive check of all executions?

STABILITY OF CONTINUOUS SYSTEMS

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0$$

with f: $\Re^n \rightarrow \Re^n$ globally Lipschitz continuous $f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$

V: $\Re^n \rightarrow \Re$ continuously differentiable (C^1) function

Rate of change of V along the execution of the ODE system:

$$\dot{V}(x) = \frac{dV(x(t))}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \frac{\partial V}{\partial x}(x) \cdot f(x)$$

(Lie derivative of V with respect to f)

$$\frac{\partial V}{\partial x} = [\frac{\partial V}{\partial x_1} \frac{\partial V}{\partial x_2} \cdots \frac{\partial V}{\partial x_n}]$$

gradient vector

STABILITY OF CONTINUOUS SYSTEMS

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(Lie derivative of V with respect to f)

$$\frac{\partial V}{\partial x} = [\frac{\partial V}{\partial x_1} \frac{\partial V}{\partial x_2} \cdots \frac{\partial V}{\partial x_n}]$$

gradient vector

No need to solve the ODE for evaluating if V(x) decreases along the executions of the system

LYAPUNOV STABILITY

Theorem (Lyapunov stability Theorem):

Let $x_e = 0$ be an equilibrium for the system and $D \subset \Re^n$ an open set containing $x_e = 0$.

If $V: D \to \Re$ is a C^1 function such that

$$V(0) = 0$$
$$V(x) > 0, \forall x \in D \setminus \{0\}$$

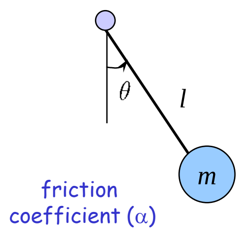
V positive definite on D

$$\dot{V}(x) \leq 0, \forall x \in D$$

V non increasing along system executions in D (negative semidefinite)

Then, x_e is stable.

EXAMPLE: PENDULUM



$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{\alpha}{m}x_2$$
$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V(x) := mgl(1-\cos(x_1)) + \frac{1}{2}mx_2^2 l^2 \geq 0$$

energy function

$$\dot{V}(x) = [mgl\sin(x_1)\ ml^2 x_2] \cdot \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = -\alpha l^2 x_2^2 \leq 0$$

x_e stable

LYAPUNOV STABILITY

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If $V: D \to \Re$ is a C^1 function such that

$$V(0) = 0$$
$$V(x) > 0, \forall x \in D \setminus \{0\}$$
$$\dot{V}(x) \leq 0, \forall x \in D$$

Then, x_e is stable.

If it holds also that

$$\dot{V}(x) < 0, \forall x \in D \setminus \{0\}$$

Then, x_e is asymptotically stable (AS)

LYAPUNOV GAS THEOREM

Theorem (Barbashin-Krasovski Theorem):

Let $x_e = 0$ be an equilibrium for the system.

If $V: \Re^n \to \Re$ is a C^1 function such that

$$V(0) = 0$$
$$V(x) > 0, \forall x \in \Re^n \setminus \{0\}$$

V positive definite on $\Re^n$

$$\dot{V}(x) < 0, \forall x \in \Re^n \setminus \{0\}$$

V decreasing along system executions in $\Re^n$ (negative definite)

$$\lim_{\|x\| \to \infty} V(x) = \infty$$

V radially unbounded

Then, x_e is globally asymptotically stable (GAS).

STABILITY OF LINEAR CONTINUOUS SYSTEMS

$$\dot{x}(t) = Ax(t)$$

- $x_e = 0$ is an equilibrium for the system

$$x(t) = e^{At}x(0),\ t \geq 0$$

$$e^{At} \rightarrow 0$$

- the elements of matrix e^{At} are linear combinations of $e^{\lambda(A)t}, te^{\lambda_i(A)t}, \ldots\ t^k e^{\lambda(A)t}$, where $\lambda(A)$ is an eigenvalue of A

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- $x_e = 0$ is an equilibrium for the system

$$x(t) = e^{At}x(0),\ t \geq 0$$

$$e^{At} \rightarrow 0$$

- $x_e = 0$ is asymptotically stable if and only if A is Hurwitz (all eigenvalues with real part <0)
- asymptotic stability ≡ GAS

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Alternative characterization…

STABILITY OF LINEAR CONTINUOUS SYSTEMS

$$\dot{x}(t) = Ax(t)$$

Theorem (necessary and sufficient condition):

The equilibrium point $x_e = 0$ is asymptotically stable if and only if for all matrices $Q = Q^T$ positive definite ($Q>0$) the

$$A^T P + PA = -Q$$ Lyapunov equation

has a unique solution $P = P^T > 0$.

Remarks:

Q positive definite ($Q>0$) iff $x^T Q x > 0$ for all $x \neq 0$

Q positive semidefinite ($Q \geq 0$) iff $x^T Q x \geq 0$ for all x and $x^T Q x = 0$ for some $x \neq 0$

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has a unique solution $P=P^T>0$.

Proof.

(if) V(x) =x^T P x is a Lyapunov function

$$\dot{V}(x) = \dot{x}^T Px + x^T P\dot{x}$$
$$= x^T(A^TP + PA)x = -x^TQx < 0, \forall x \neq 0$$

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Proof.

(only if) Consider $P = \int_0^\infty e^{A^Tt}Qe^{At}dt$

$$A^TP + PA = \int_0^\infty A^Te^{A^Tt}Qe^{At}dt + \int_0^\infty e^{A^Tt}Qe^{At}Adt$$
$$= \int_0^\infty \frac{d}{dt}e^{A^Tt}Qe^{At}dt = -Q$$

STABILITY OF LINEAR CONTINUOUS SYSTEMS

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has a unique solution $P=P^T$ >0.

Proof.

(only if) Consider $P = \int_0^\infty e^{A^T t} Q e^{At} dt$

P = P^T and P>0 easy to show

P unique can be proven by contradiction

STABILITY OF LINEAR CONTINUOUS SYSTEMS

$$\dot{x}(t) = Ax(t)$$

Remarks: for a linear system

- existence of a (quadratic) Lyapunov function $V(x) = x^T P x$ is a necessary and sufficient condition for asymptotic stability
- it is easy to compute a Lyapunov function since the Lyapunov equation

$$A^TP+PA = -Q$$

is a linear algebraic equation

STABILITY OF LINEAR CONTINUOUS SYSTEMS

$$\dot{x}(t) = Ax(t)$$

Theorem (exponential stability):

Let the equilibrium point $x_e = 0$ be asymptotically stable. Then, the rate of convergence to $x_e = 0$ is exponential:

$$\|x(t)\| \leq \mu e^{-\lambda_0 t}\|x_0\|, t \geq 0$$

for all $x(0) = x_0 \in \Re^n$, where $\lambda_0 \in (0, \min_i |\mathrm{Re}\{\lambda_i(A)\}|)$ and $\mu > 0$ is an appropriate constant.

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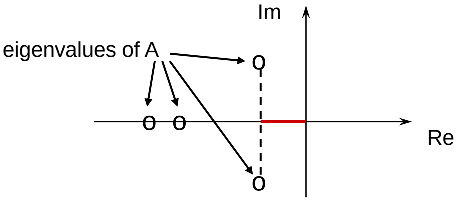

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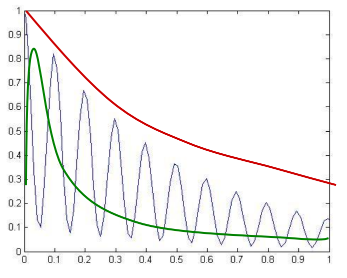

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for all x(0) = $x_0 \in \Re^n$, where $\lambda_0 \in (0, \min_i |Re\{\lambda_i(A)\}|)$ and μ >0 is an appropriate constant.

Remark:
$$\|x(t)\| = \|e^{At}x_0\| \le \mu e^{-\lambda_0 t}\|x_0\|, t \ge 0, \forall x_0$$
$$\rightarrow \|e^{At}\| = \sup_{x_0 \ne 0} \frac{\|e^{At}x_0\|}{\|x_0\|} \le \mu e^{-\lambda_0 t}, t \ge 0$$

STABILITY OF LINEAR CONTINUOUS SYSTEMS

$$\dot{x}(t) = Ax(t)$$

Proof (exponential stability):

A + λ_0 I is Hurwitz (eigenvalues are equal to λ(A) + λ_0)

Then, there exists P = P^T >0 such that

$$(A + \lambda_0 I)^T P + P (A + \lambda_0 I) < 0$$

which leads to

$$x(t)^T [A^T P + P A] x(t) < - 2 \lambda_0 x(t)^T P x(t)$$

Define V(x) = x^T P x, then

$$\dot{V}(x(t)) < -2\lambda_0 V(x(t))$$

from which

$$V(x(t)) < e^{-2\lambda_0 t} V(x_0)$$

STABILITY OF LINEAR CONTINUOUS SYSTEMS

$$\dot{x}(t) = Ax(t)$$

(cont'd) Proof (exponential stability):

$$x^T \lambda_{\min}(P) I x \leq V(x) = x^T P x \leq x^T \lambda_{\max}(P) I x$$

$$\lambda_{\min}(P) \|x(t)\|^2 \leq V(x(t)) < e^{-2\lambda_0 t} V(x_0) \leq e^{-2\lambda_0 t} \lambda_{\max}(P) \|x_0\|^2$$

thus finally leading to

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\lambda_0 t} \|x_0\|$$

STABILITY OF LINEAR CONTINUOUS SYSTEMS

$$\dot{x}(t) = Ax(t)$$

- $x_e = 0$ is an equilibrium for the system
- $x_e = 0$ is asymptotically stable if and only if A is Hurwitz (all eigenvalues with real part <0)
- asymptotic stability ≡ GAS ≡ exponential stability ≡ GES