POLITECNICO DI MILANO

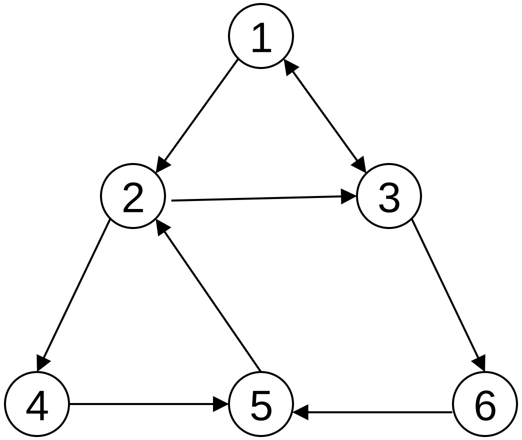
Math tools: basics on graph theory

Maria Prandini

Outline

- Modeling agent-to-agent communication as a graph
- Undirected and directed graph
- Connectivity and strong connectivity
- Application to distributed averaging
- Extension to the time-varying case

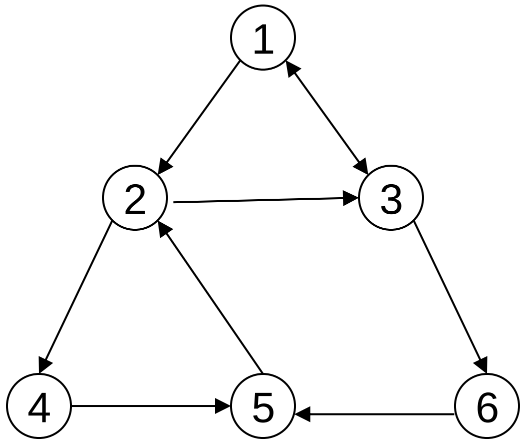
Modeling



agent $\rightarrow$ node i

agent j communicates with agent i $\rightarrow$ edge $e = (j, i)$

Modeling



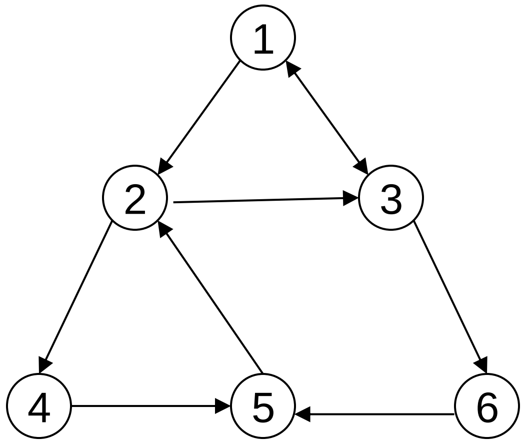
agent $\rightarrow$ node i

agent j communicates with agent i $\rightarrow$ edge $e = (j, i)$

Graph modeling the agents communication: $G = (V, E)$

$$V = \{1,2,\dots,m\} \quad E = \{(j,i)\colon\ j \text{ communicates with } i\} \subseteq V \times V$$

Modeling



agent $\rightarrow$ node i

agent j communicates with agent i $\rightarrow$ edge $e = (j, i)$

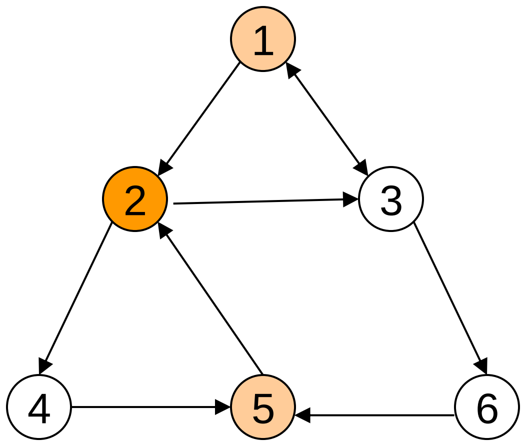
Graph modeling the agents communication: $G = (V, E)$

$$V = \{1, 2, \ldots, m\} \quad E = \{(j, i) \colon j \text{ communicates with } i\} \subseteq V \times V$$

Neighbors of agent i:

$$N_i = \{j \colon (j, i) \in E\}$$

Modeling



$N_2 = \{1,5\}$

agent $\rightarrow$ node i

agent j communicates with agent i $\rightarrow$ edge $e = (j, i)$

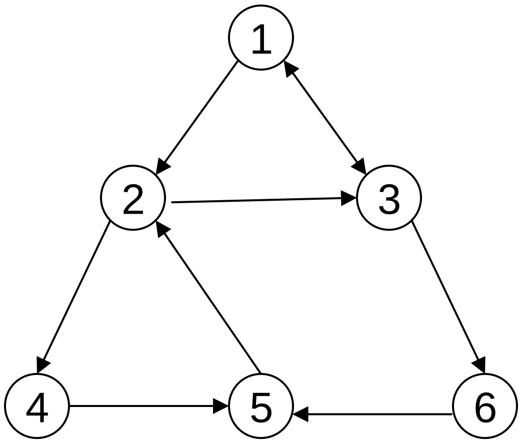
Graph modeling the agents communication: $G = (V, E)$

$$V = \{1,2,\ldots,m\} \;\; E = \{(j,i)\colon\, j \text{ communicates with } i\} \subseteq V \times V$$

Neighbors of agent i:

$$N_i = \{j\colon (j,i) \in E\}$$

Path



agent $\rightarrow$ node i

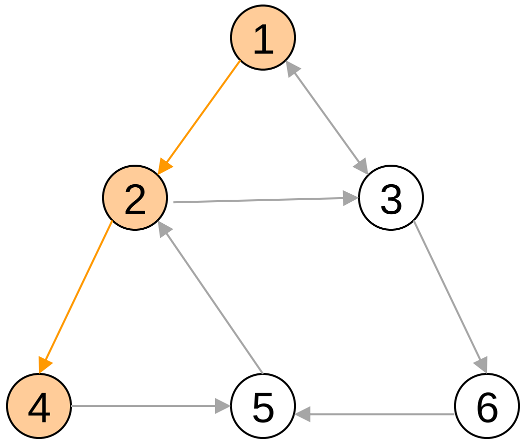
agent j communicates with agent i $\rightarrow$ edge $e = (j, i)$

Graph modeling the agents communication: $G = (V, E)$

Path is a subgraph $\pi = (V_\pi, E_\pi) \subset G$ with

$$V_\pi = \{i_1, i_2, \dots, i_k\} \subset \mathrm{V} \qquad E_\pi = \{(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)\} \subset E$$

Path



example of a path of length 2

agent $\rightarrow$ node i

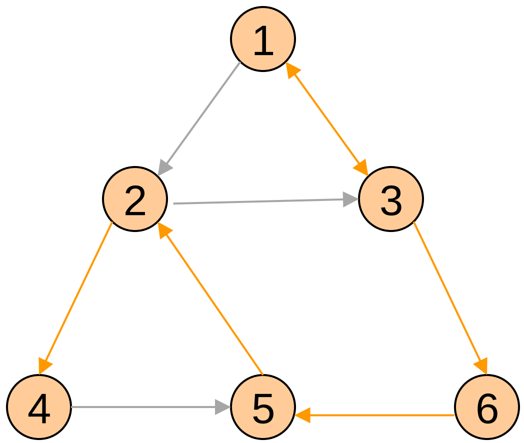
agent j communicates with agent i $\rightarrow$ edge $e = (j, i)$

Graph modeling the agents communication: $G = (V, E)$

Path is a subgraph $\pi = (V_\pi, E_\pi) \subset G$ with

$$V_\pi = \{i_1, i_2, \dots, i_k\} \subset \mathrm{V} \qquad E_\pi = \{(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)\} \subset E$$

Path



example of a path of length 5

agent $\rightarrow$ node i

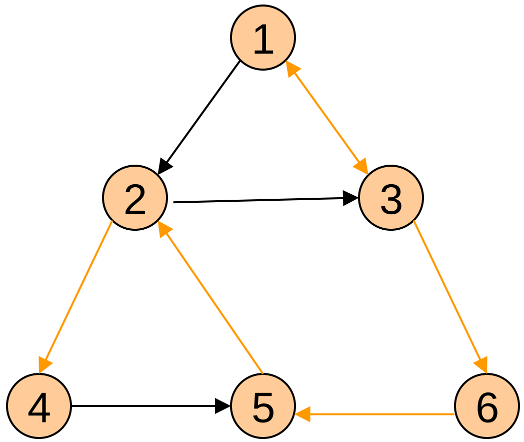
agent j communicates with agent i $\rightarrow$ edge $e = (j, i)$

Graph modeling the agents communication: $G = (V, E)$

Path is a subgraph $\pi = (V_\pi, E_\pi) \subset G$ with

$$V_\pi = \{i_1, i_2, \dots, i_k\} \subset \mathrm{V} \qquad E_\pi = \{(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)\} \subset E$$

Path



example of a path of length 5

$\downarrow$

shortest path defines the distance between two nodes

agent $\rightarrow$ node i

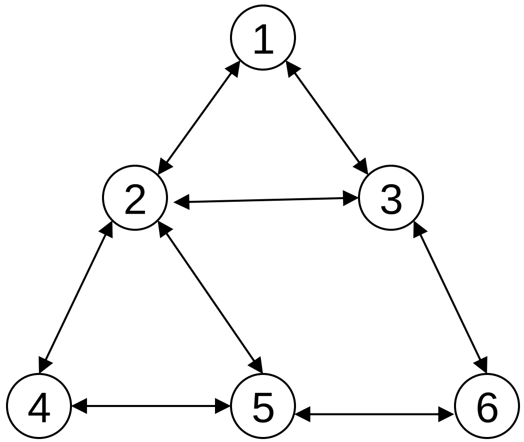
agent j communicates with agent $i \rightarrow$ edge $e = (j, i)$

Graph modeling the agents communication: $G = (V, E)$

Path is a subgraph $\pi = (V_\pi, E_\pi) \subset G$ with

$$V_\pi = \{i_1, i_2, \dots, i_k\} \subset \mathrm{V} \qquad E_\pi = \{(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)\} \subset E$$

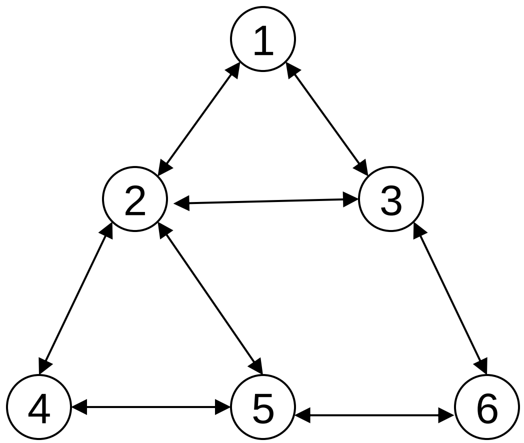
Undirected graph and connectivity



graph $G = (V, E)$ is undirected if

$$(j, i) \in E \Rightarrow (i, j) \in E$$

Undirected graph and connectivity



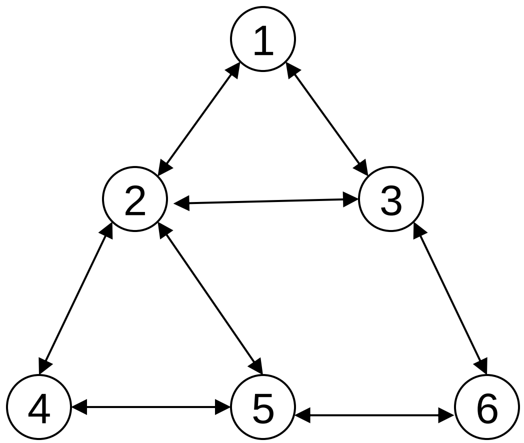
graph $G = (V, E)$ is undirected if

$$(j, i) \in E \Rightarrow (i, j) \in E$$

an undirected graph is connected,

if there exists a path π between any two distinct nodes

Undirected graph and connectivity



graph $G = (V, E)$ is undirected if

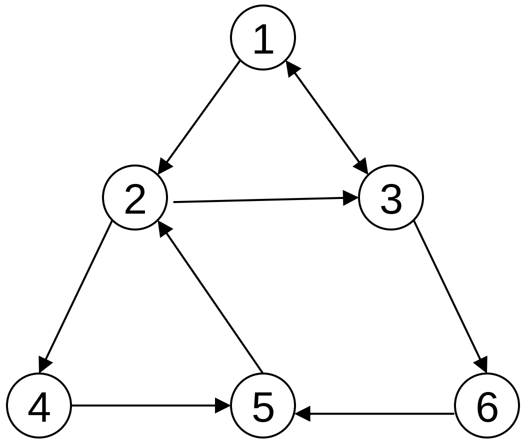
$$(j, i) \in E \Rightarrow (i, j) \in E$$

an undirected graph is connected,

if there exists a path π between any two distinct nodes

diameter of a connected graph is the maximum distance between two nodes

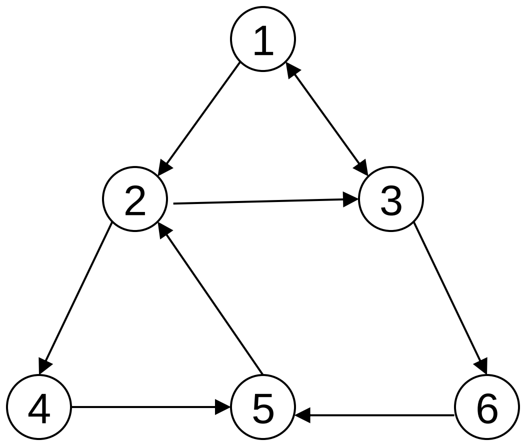
Directed graph and connectivity



graph $G = (V, E)$ is directed if

$$\exists (j, i) \in E \text{ such that } (i, j) \notin E$$

Directed graph and connectivity



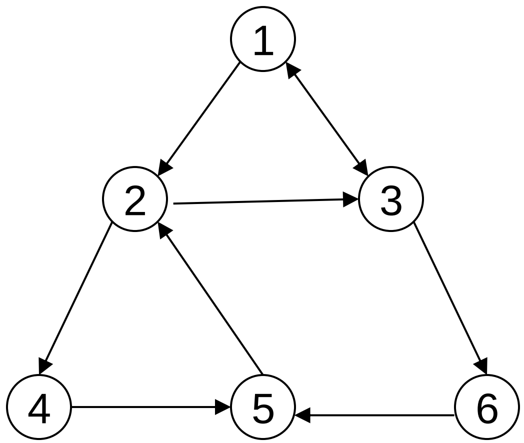
graph $G = (V, E)$ is directed if

$$\exists (j, i) \in E \text{ such that } (i, j) \notin E$$

a directed graph is strongly connected,
if there exists a directed path π between any two distinct nodes

a directed graph is weakly connected,
if the undirected graph obtained by making all arches bidirectional is connected

Directed graph and connectivity



graph $G = (V, E)$ is directed if

$$\exists (j, i) \in E \text{ such that } (i, j) \notin E$$

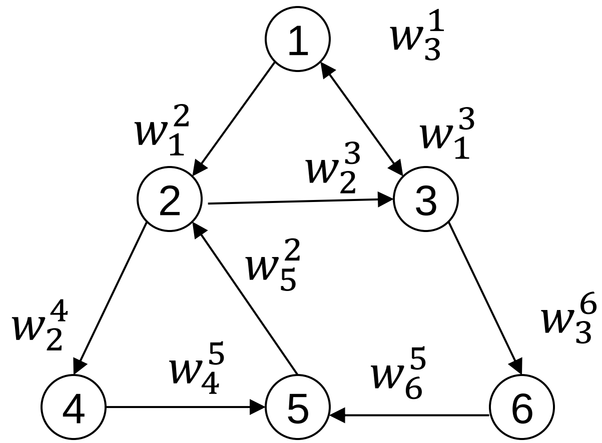
a directed graph is strongly connected,

if there exists a directed path π between any two distinct nodes

a directed graph is weakly connected,

if the undirected graph obtained by making all arches bidirectional is connected

Weighted graph



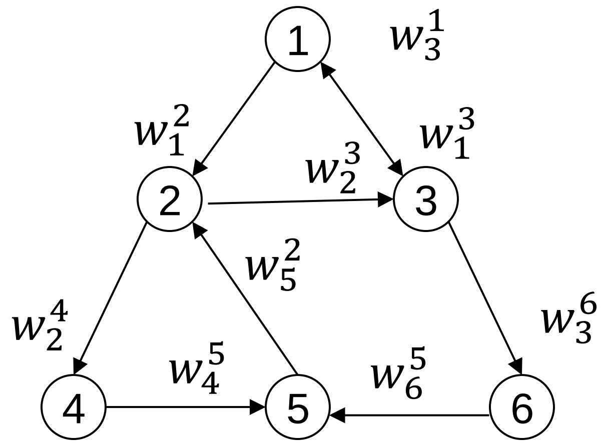
a weighted graph is a graph $G = (V, E)$ together with

a map $\varphi: E \rightarrow R$ that assigns a weight $w_j^i = \varphi((j, i))$ to an edge $(j, i) \in E$

We can then define the $m \times m$ weight matrix W such that

$$W(i, j) = \begin{cases} w_j^i & (j, i) \in E \\ 0 & \text{otherwise} \end{cases}$$

Weighted graph



W is row-stochastic if $W(i,j) \geq 0$ and $\sum_j W(i,j) = 1, i = 1, \ldots, m$

W is column-stochastic if $W(i,j) \geq 0$ and $\sum_i W(i,j) = 1, j = 1, \ldots, m$

W is doubly-stochastic if it is both row and column stochastic

If $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, then, $W\mathbf{1} = \mathbf{1}$ (row-stochastic) and $\mathbf{1}^T W = \mathbf{1}^T$ (column-stochastic)

Application: consensus protocols

- m agents communicate along a set of links described by a graph $G = (V, E)$ (by definition each agent communicates with itself, and hence belongs to its neighborhood)
- each agent i has a scalar state x_i with initial value $x_i(0)$ and the agents aim at jointly reaching consensus to the average of their initial states

$$\bar{x}(0) = \frac{1}{m} \sum_i x_i(0)$$

Application: consensus protocols

Distributed averaging solution:

- Associate a weight $W(i,j) > 0$ to $(j,i) \in E$
- Let each agent compute in parallel the weighted average

$$x_i(k+1) = \sum_{j \in N_i} W(i,j)\, x_j(k)$$

Application: consensus protocols

Distributed averaging solution:

- Associate a weight $W(i,j) > 0$ to $(j,i) \in E$
- Let each agent compute in parallel the weighted average

$$x_i(k+1) = \sum_{j \in N_i} W(i,j)\, x_j(k)$$

Theorem

If the weight matrix W is doubly-stochastic and the communication graph $G = (V, E)$ is (strongly) connected, then

$$\lim_{k \to \infty} x_i(k) = \bar{x}(0), i = 1, \dots, m$$

Proof of the distributed average consensus theorem

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ be the collection the states of the m agents. Then, we have

$$x(k+1) = Wx(k)$$

Application: consensus protocols

Proof of the distributed average consensus theorem

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ be the collection the states of the m agents. Then, we have

$$x(k+1) = Wx(k)$$

Property 1: the average $\bar{x}(k) = \frac{1}{m}\sum_i x_i(k)$ is preserved throughout iterations

Since $\mathbf{1}^T W = \mathbf{1}^T$, we have that

$$\bar{x}(k+1) = \frac{1}{m}\mathbf{1}^T x(k+1) = \frac{1}{m}\mathbf{1}^T W x(k) = \frac{1}{m}\mathbf{1}^T x(k) = \bar{x}(k) = \bar{x}(0)$$

Application: consensus protocols

Proof of the distributed average consensus theorem

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ be the collection the states of the m agents. Then, we have

$$x(k+1) = Wx(k)$$

Property 1: $$\bar{x}(k) = \frac{1}{m}\mathbf{1}^T x(k) = \bar{x}(0), k > 0$$

Note also that since

- $\mathrm{W}\mathbf{1}\bar{x}(k) = \mathbf{1}\bar{x}(k)$
- $\frac{1}{m}\mathbf{1}\mathbf{1}^T\big(x(k) - \mathbf{1}\bar{x}(k)\big) = \mathbf{1}(\frac{1}{m}\mathbf{1}^T x(k) - \frac{1}{m}\mathbf{1}^T\mathbf{1}\bar{x}(k)) = 0$

Proof of the distributed average consensus theorem

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ be the collection the states of the m agents. Then, we have

$$x(k+1) = Wx(k)$$

Property 1: $$\bar{x}(k) = \frac{1}{m}\mathbf{1}^T x(k) = \bar{x}(0), k > 0$$

Note also that since

- W$\mathbf{1}\bar{x}(k) = \mathbf{1}\bar{x}(k)$
- $\frac{1}{m}\mathbf{1}\mathbf{1}^T\big(x(k) - \mathbf{1}\bar{x}(k)\big) = \mathbf{1}(\frac{1}{m}\mathbf{1}^T x(k) - \frac{1}{m}\mathbf{1}^T\mathbf{1}\bar{x}(k)) = 0$

the dynamics of the disagreement vector is given by:

$$\begin{aligned} x(k+1) - \mathbf{1}\bar{x}(k+1) &= Wx(k) - \mathbf{1}\bar{x}(k) = W(x(k) - \mathbf{1}\bar{x}(k)) \\ &= \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k)) \end{aligned}$$

Application: consensus protocols

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$.

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$.

Preliminary results:

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$.

Preliminary results:

- since W is doubly-stochastic, then, it has all eigenvalues with $|\lambda| \leq 1$

$$\lambda v = Wv \rightarrow |\lambda| \max_i |v_i| = |\lambda||v_{i^*}| = |\textstyle\sum_j W(i^*,j)v_j| \leq \textstyle\sum_j W(i^*,j)|v_j| \leq |v_{i^*}|$$

Application: consensus protocols

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$

Preliminary results:

- since W is doubly-stochastic, then, it has all eigenvalues with $|\lambda| \leq 1$

$$\lambda v = Wv \rightarrow |\lambda| \max_i |v_i| = |\lambda||v_{i^*}| = |\textstyle\sum_j W(i^*,j)v_j| \leq \textstyle\sum_j W(i^*,j)|v_j| \leq |v_{i^*}|$$

- each eigenvector v associated to an eigenvalue with $|\lambda| < 1$ is orthogonal to $\mathbf{1}$

$$\mathbf{1}^T W v = \mathbf{1}^T \lambda v \rightarrow \mathbf{1}^T v = \lambda \mathbf{1}^T v \rightarrow \mathbf{1}^T v = 0$$

Application: consensus protocols

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$

Preliminary results:

- since W is doubly-stochastic, then, it has all eigenvalues with $|\lambda| \leq 1$

$$\lambda v = Wv \rightarrow |\lambda| \max_i |v_i| = |\lambda||v_{i^*}| = |\textstyle\sum_j W(i^*, j)v_j| \leq \sum_j W(i^*, j)|v_j| \leq |v_{i^*}|$$

- each eigenvector v associated to an eigenvalue with $|\lambda| < 1$ is orthogonal to $\mathbf{1}$

$$\mathbf{1}^T W v = \mathbf{1}^T \lambda v \rightarrow \mathbf{1}^T v = \lambda \mathbf{1}^T v \rightarrow \mathbf{1}^T v = 0$$

- $\lambda = 1$ is an eigenvalue of W and $\mathbf{1}$ is an eigenvector associated with it

$$W\mathbf{1} = \mathbf{1} \rightarrow \lambda = 1$$

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$

Preliminary results:

- since W is doubly-stochastic, then, it has all eigenvalues with $|\lambda| \leq 1$

  $$\lambda v = Wv \rightarrow |\lambda| \max_i |v_i| = |\lambda||v_{i^*}| = |\textstyle\sum_j W(i^*, j) v_j| \leq \textstyle\sum_j W(i^*, j)|v_j| \leq |v_{i^*}|$$

- each eigenvector v associated to an eigenvalue with $|\lambda| < 1$ is orthogonal to $\mathbf{1}$

  $$\mathbf{1}^T W v = \mathbf{1}^T \lambda v \rightarrow \mathbf{1}^T v = \lambda \mathbf{1}^T v \rightarrow \mathbf{1}^T v = 0$$

- $\lambda = 1$ is an eigenvalue of W and $\mathbf{1}$ is an eigenvector associated with it

  $$W\mathbf{1} = \mathbf{1} \rightarrow \lambda = 1$$

- since the graph is strongly connected, by Perron–Frobenius theorem, only one eigenvalues of W satisfy $|\lambda| = 1$, it is equal to 1 and simple with eigenvector $\mathbf{1}$

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$

To this purpose, by using the preliminary results we shall show that

1. the eigenvalue $\lambda = 1$ of W with eigenvector $\mathbf{1}$ is shifted to 0
2. all the other eigenvalues of W (with $|\lambda| < 1$) are preserved

Application: consensus protocols

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$.

To this purpose, by using the preliminary results we shall show that

1. the eigenvalue $\lambda = 1$ of W with eigenvector $\mathbf{1}$ is shifted to 0

$$\left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)\mathbf{1} = W\mathbf{1} - \mathbf{1}\frac{1}{m}\mathbf{1}^T\mathbf{1} = \mathbf{1} - \mathbf{1} = 0 \cdot \mathbf{1}$$

Proof of the distributed average consensus theorem

$$x(k+1) - \mathbf{1}\bar{x}(k+1) = \left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)(x(k) - \mathbf{1}\bar{x}(k))$$

We now just need to prove that $W - \frac{1}{m}\mathbf{1}\mathbf{1}^T$ has all eigenvalues with $|\lambda| < 1$.

To this purpose, by using the preliminary results we shall show that

1. the eigenvalue $\lambda = 1$ of W with eigenvector $\mathbf{1}$ is shifted to 0
2. all the other eigenvalues of W (with $|\lambda| < 1$) are preserved

Let v be an eigenvector associated with an eigenvalue λ of W with $|\lambda| < 1$, then, it is orthogonal to $\mathbf{1}$ and, hence,

$$\left(W - \frac{1}{m}\mathbf{1}\mathbf{1}^T\right)v = Wv - \frac{1}{m}\mathbf{1}\mathbf{1}^T v = Wv = \lambda v$$

Time-varying setting

Let the time-varying graphs $G_k = (V, E_k), k = 0,1, \dots$ with

$$V = \{1,2, \dots, m\} \;\; E_k = \{(j, i)\colon\ j \text{ communicates with } i \text{ at time } k\}$$

model a time-varying communication network

Introduce the weight matrices $W_k, \mathrm{k} = 0,1, \dots$ associated with $G_k, k = 0,1, \dots$ and consider the distributed algorithm

$$x(k+1) = W_k x(k)$$

Time-varying setting

Assumptions:

- Connectivity

  (V, E_∞) strongly connected where
  $$E_\infty = \{(j,i)\colon\ j \text{ communicates with } i \text{ infinitely often}\}$$
- Bounded intercommunication time (partial asynchronism)

  there exists $T \geq 1$ such that for every $(j,i) \in E_\infty$,
  $$(j,i) \in E_k \cup E_{k+1} \cup \cdots \cup E_{k+T-1}, k \geq 0$$
  i.e., agent *i* receives information from a neighboring agent *j* at least once every consecutive *T* iterations

- Weights rules

  each W_k is doubly-stochastic and there exists $\eta > 0$ such that
  $$W_k(i,i) \geq \eta, \forall i, \forall k$$
  $$W_k(i,j) \geq \eta, \forall (j,i) \in E_k$$

Time-varying setting

Let the time-varying graphs $G_k = (V, E_k), k = 0,1, \ldots$ with

$$V = \{1,2,\ldots,m\} \;\; E_k = \{(j,i)\colon \; j \text{ communicates with } i \text{ at time } k\}$$

model a time-varying communication network

Introduce the weight matrices $W_k, \mathrm{k} = 0,1, \ldots$ associated with $G_k, k = 0,1, \ldots$ and consider the distributed algorithm

$$x(k+1) = W_k x(k)$$

Theorem

Under the previous assumptions on the communication network and the weights, the agents asymptotically reach consensus on the average

$$\lim_{k\to\infty} x_i(k) = \bar{x}(0), i = 1, \ldots, m$$