Variable Structure Control (VSC)

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Lecture 3

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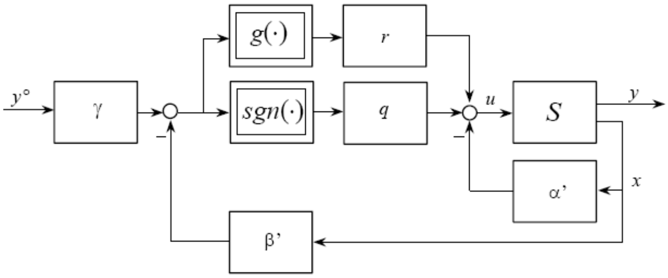
LATEST LECTURE

- Description of an approach to the design of variable structure controllers for output regulation of a linear system
- Evaluation of the performance of the controller on a numerical example
- Robustness of the control strategy with respect to parameter uncertainty

LATEST LECTURE



LATEST LECTURE

System S:

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_1\, s^{n\text{-}1} + b_2\, s^{n\text{-}2} + \ldots + b_n}{s^n + a_1\, s^{n\text{-}1} + a_2\, s^{n\text{-}2} + \ldots + a_n}$$

Key ingredients in the design of the controller:

- the switching function $s(x) = \beta' x - \gamma y°$ with $\gamma = \beta_{n-1}/b_n$ and $\beta' = [\beta_{n-1}\ \beta_{n-2}\ \ldots \beta_1\ 1]$

LATEST LECTURE

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- the switching function $s(x) = \beta' x - \gamma y°$ with $\gamma = \beta_{n-1}/b_n$ and $\beta' = [\beta_{n-1}\ \beta_{n-2}\ \ldots \beta_1\ 1]$, which defines:
  - the sliding surface $s(x) = 0$
  - the a.s. characteristic polynomial of the (n-1)-dimensional system S* governing the state evolution over the sliding surface:
    $$\chi^*(\lambda) = \lambda^{n-1} + \beta_1 \lambda^{n-1} + \cdots + \beta_{n-1}$$
  - the value $\bar{y}$ for the system output at the equilibrium over the sliding surface: $\bar{y} = y°$

LATEST LECTURE

System S:

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- the switching function $s(x) = \beta' x - \gamma y°$ with $\gamma = \beta_{n-1}/b_n$ and $\beta' = [\beta_{n-1}\ \beta_{n-2}\ \ldots \beta_1\ 1]$
- the control input
  $$u = -\alpha' x + q\ sgn(\gamma y° - \beta' x) + \mathrm{r}\ \mathrm{g}(\gamma y° - \beta' x)$$
  with $q > 0, r \geq 0$, $sg(s) > 0, \alpha' = \beta' \mathrm{A}$, where A is the matrix of S in controllable form

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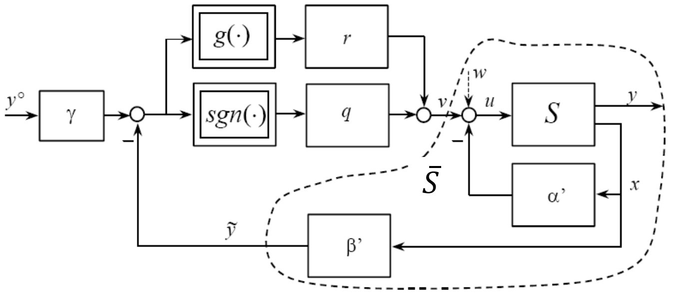
Key ingredients in the design of the controller:

- the switching function $s(x) = \beta' x - \gamma y^\circ$ with $\gamma = \beta_{n-1}/b_n$ and $\beta' = [\beta_{n-1}\ \beta_{n-2}\ \ldots \beta_1\ 1]$
- the control input
$$u = -\alpha' x + q\, sgn(\gamma y^\circ - \beta' x) + \mathrm{r}\, \mathrm{g}(\gamma y^\circ - \beta' x)$$
with $q > 0, r \geq 0, sg(s) > 0, \alpha' = \beta' \mathrm{A}$, where A is the matrix of S in controllable form, which makes system S reach the sliding surface in finite time $t_r \leq |s(x(0))|/q$

ROBUSTNESS W.R.T. LOAD DISTURBANCE

Suppose that S is subject to some load disturbance $w(t)$



ROBUSTNESS W.R.T. LOAD DISTURBANCE

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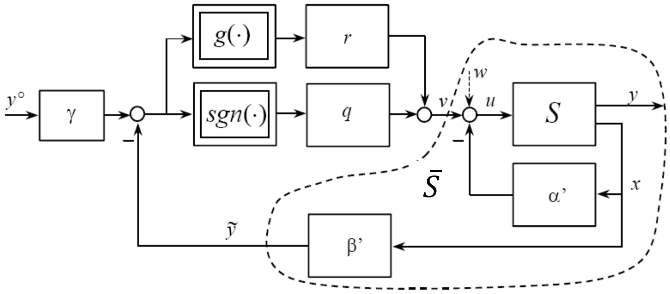
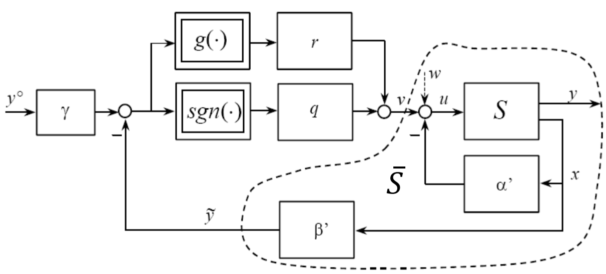


Statement:

If $|w(t)| \leq W < q, \forall t$, then:
the state will reach the sliding surface $s(x) = 0$ in finite time and will evolve according to the dynamics of S*.
Only the time to convergence is affected by w

ROBUSTNESS W.R.T. LOAD DISTURBANCE



Sub-system $\bar{S}$

$$\bar{S} : \begin{cases} \dot{x} = A\,x + B\,u \\ \tilde{y} = \beta'\,x \end{cases} \quad , \qquad u = v + w - \alpha'\,x$$

$$\Downarrow$$

$$\bar{S}' : \begin{cases} \dot{x} = \tilde{A}\,x + B\,(v + w) \\ \tilde{y} = \beta'\,x \end{cases} \qquad \check{A} = A - B\alpha'$$

ROBUSTNESS W.R.T. LOAD DISTURBANCE

$$\bar{S}: \quad \begin{cases} \dot{x} = \tilde{A}\, x + B\,(v + w) \\ \tilde{y} = \beta'\, x \end{cases} \qquad \check{A} = A - B\alpha'$$

$$\alpha' := \beta' A = [-a_n \quad \beta_{n-1} - a_{n-1} \quad \beta_{n-2} - a_{n-2} \quad \ldots \quad \beta_1 - a_1]$$

$$\beta' := [\beta_{n-1} \quad \beta_{n-2} \quad \ldots \quad \beta_1 \quad 1]$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{bmatrix}$$

ROBUSTNESS W.R.T. LOAD DISTURBANCE

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$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{bmatrix}$$

ROBUSTNESS W.R.T. LOAD DISTURBANCE

$$\bar{S}: \begin{cases} \dot{x} = \tilde{A}\, x + B\,(v + w) \\ \tilde{y} = \beta'\, x \end{cases} \qquad \check{A} = A - B\alpha'$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & -\beta_{n-1} & -\beta_{n-2} & \dots & -\beta_2 & -\beta_1 \end{bmatrix}$$

ROBUSTNESS W.R.T. LOAD DISTURBANCE

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$$\beta' := [\beta_{n-1} \quad \beta_{n-2} \quad \dots \quad \beta_1 \quad 1]$$

$$\tilde{G}(s) = \frac{s^{n-1} + \beta_1\, s^{n-2} + \dots + \beta_{n-1}}{s^n + \beta_1\, s^{n-1} + \beta_2\, s^{n-2} + \dots + \beta_{n-1}\, s} = \frac{\chi^*(s)}{s\,\chi^*(s)} = \frac{1}{s}$$

ROBUSTNESS W.R.T. LOAD DISTURBANCE

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System $\bar{S}$:

- externally $v + w \rightarrow \tilde{y}$ behaves like an integrator

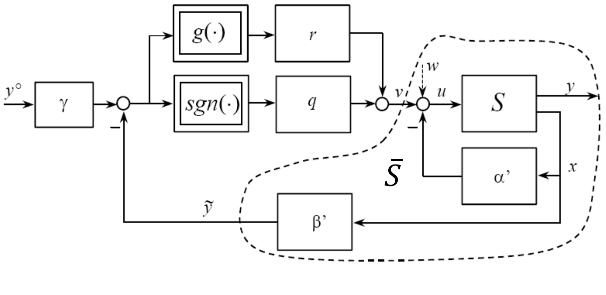
$$\dot{\tilde{y}} = v + w$$

- has hidden dynamics with characteristic polynomial $\chi^*(s)$

ROBUSTNESS W.R.T. LOAD DISTURBANCE

$$\dot{\tilde{y}} = v + w$$

If $w = 0$ $\quad \dot{\tilde{y}} = q\, sgn(\gamma\, y° - \tilde{y}) + r\, g(\gamma\, y° - \tilde{y})$



ROBUSTNESS W.R.T. LOAD DISTURBANCE

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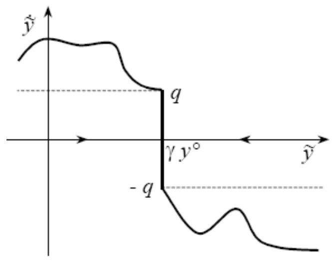

The (pseudo)-equilibrium $\breve{y} = \gamma y^{\circ}$ is reached in finite time.

ROBUSTNESS W.R.T. LOAD DISTURBANCE

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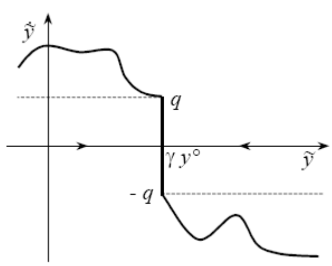


The (pseudo)-equilibrium $\breve{y} = \gamma y^{\circ}$ is reached in finite time.

If $|w(t)| \leq W < q, \forall t$, then, the sign of the derivative of $\breve{y}$ is preserved, so that $\breve{y} = \gamma y^{\circ}$ is still reached in finite time.
The time to reach $\breve{y} = \gamma y^{\circ}$ will however depend on w.

ROBUSTNESS W.R.T. LOAD DISTURBANCE

When $\check{y} = \gamma y°$, then,

$$\check{y} = \beta' x = \gamma y°$$

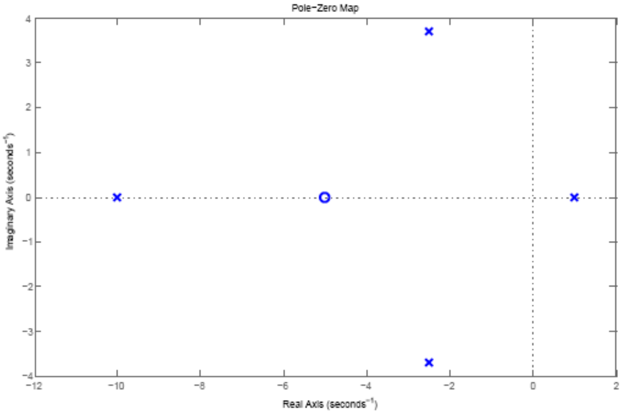
Thus, $s(x) = \beta' x - \gamma y° = 0$ and

→ the system evolves on the sliding surface according to the dynamics of S*;

→ the output y tends to $y°$

STATE NOT AVAILABLE: A NUMERICAL EXAMPLE

$$G(s) = \frac{-400(s+5)}{(s^2+5s+20)(s+10)(s-1)}$$
$$= \frac{-400(s+5)}{s^4+14s^3+55s^2+130s-200}$$



STATE NOT AVAILABLE: A NUMERICAL EXAMPLE

$$G(s) = \frac{-400(s+5)}{s^4 + 14s^3 + 55s^2 + 130s - 200}$$

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 200 & -130 & -55 & -14 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = -[2000 \quad 400 \quad 0 \quad 0\ ]$$

$a_1 = 14,\quad a_2 = 55,\quad a_3 = 130,\quad a_4 = -200$

$b_1 = b_2 = 0,\quad b_3 = -400,\quad b_4 = -2000.$

STATE NOT AVAILABLE: A NUMERICAL EXAMPLE

Recall that the β coefficients of the switching function [n=4]

$$s(x) := \beta_{n-1} x_1 + \beta_{n-2} x_2 + \ldots + \beta_1 x_{n-1} + x_n - \bar{w}, \quad \bar{w} = \gamma\, y^\circ$$

appear in the characteristic polynomial of the linear system S* of order 3 governing the dynamics of S when restricted to the sliding surface $s(x) = 0$:

$$\chi^*(\lambda) = \lambda^{n-1} + \beta_1 \lambda^{n-2} + \ldots + \beta_{n-1}$$

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appear in the characteristic polynomial of the linear system S* of order 3 governing the dynamics of S when restricted to the sliding surface $s(x) = 0$:

$$\chi^*(\lambda) = \lambda^{n-1} + \beta_1 \lambda^{n-2} + \ldots + \beta_{n-1}$$

Set the eigenvalues of S* equal to those of S that are stable:

$$\chi^*(\lambda) = (\lambda^2 + 5\,\lambda + 20)(\lambda + 10) = \lambda^3 + 15\,\lambda^2 + 70\,\lambda + 200$$

So that $\beta' := [\beta_3 \quad \beta_2 \quad \beta_1 \quad 1] = [200 \quad 70 \quad 15 \quad 1]$

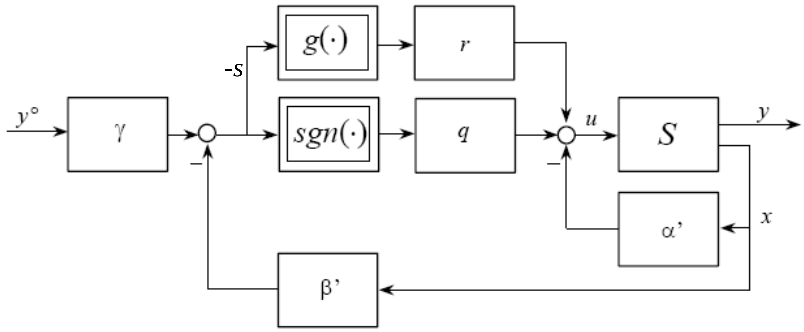
$\gamma = \beta_3/b_4 = -\,0.1 \qquad \alpha' := \beta' A = [200 \quad 70 \quad 15 \quad 1]$

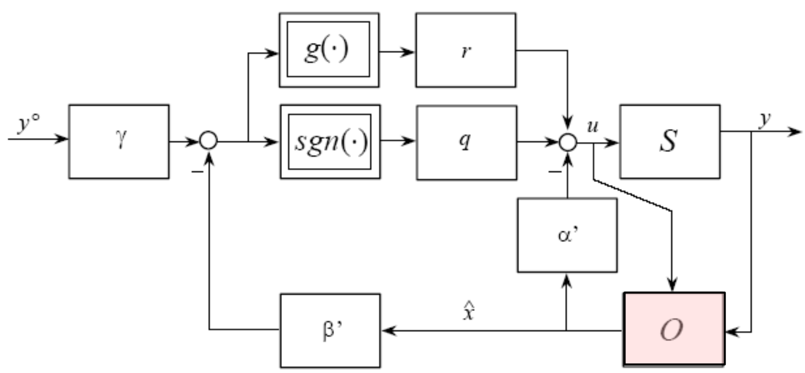
STATE NOT AVAILABLE: A NUMERICAL EXAMPLE

$$G(s) = \frac{-400(s+5)}{s^4 + 14s^3 + 55s^2 + 130s - 200}$$

Since the state is not directly measurable, we resort to the asymptotic state observer (Luenberger observer) and use $\hat{x}$ in place of x in the sliding mode control law:

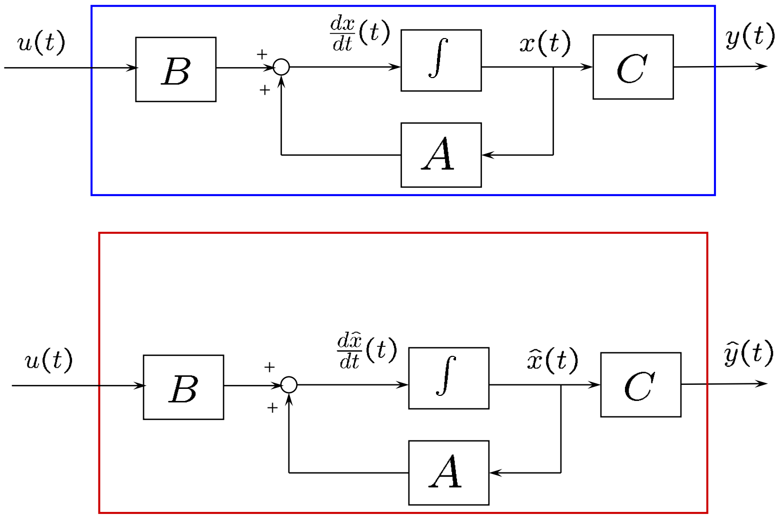
$$\mathrm{u} = -\alpha'\hat{x} + q\, sgn(\gamma y^\circ - \beta'\hat{x}) + rg(\gamma y^\circ - \beta'\hat{x})$$



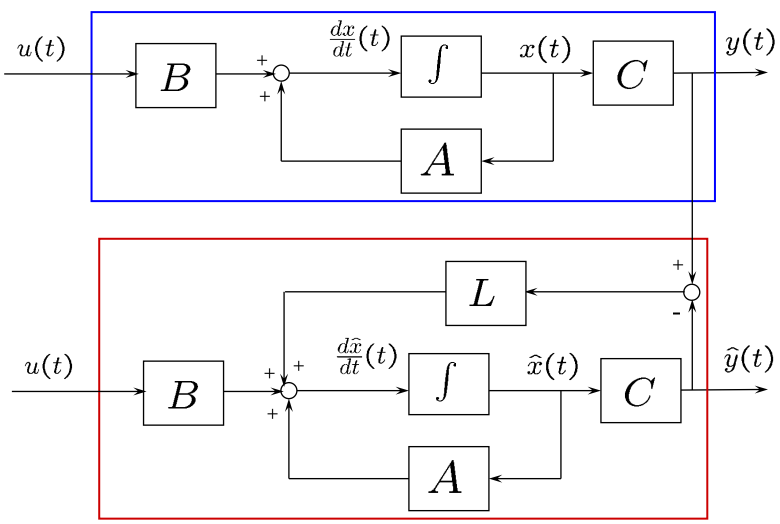


$$u = -\alpha'\hat{x} + q\ sgn(\gamma y^{\circ} - \beta'\hat{x}) + rg(\gamma y^{\circ} - \beta'\hat{x})$$

ASYMPTOTIC OBSERVER



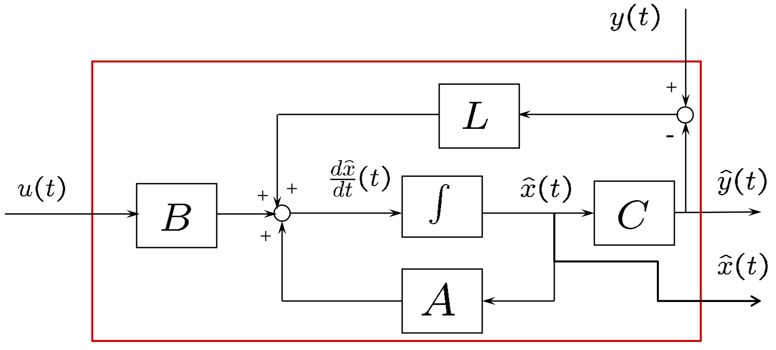
ASYMPTOTIC OBSERVER



ASYMPTOTIC OBSERVER

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \boxed{L}(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = C\hat{x}(t)$$

observer gain



DYNAMICS OF THE STATE ESTIMATION ERROR

$$e(t) := x(t) - \hat{x}(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\,(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = C\hat{x}(t)$$

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$$y(t) = Cx(t)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left(Cx(t) - C\hat{x}(t)\right)$$
$$\hat{y}(t) = C\hat{x}(t)$$

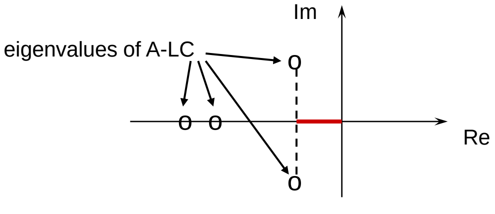
$$\boxed{\dot{e}(t) = (A - LC)\, e(t)}$$

ASYMPTOTIC OBSERVER

$$\boxed{\dot{e}(t) = (A - LC)\, e(t)}$$

If (A,C) is observable, then, L can be designed so that A-LC has arbitrarily chosen eigenvalues and the estimation error converges exponentially to zero with rate $\lambda_0 \in (0, \min_i |\mathrm{Re}\{\lambda_i(A\text{-}LC)\}|)$

$$\|e(t)\| \leq \mu e^{-\lambda_0 t} \|e(0)\|,\ t \geq 0, \quad \forall e(0) = e_0 \in \Re^n$$



STATE NOT AVAILABLE: A NUMERICAL EXAMPLE

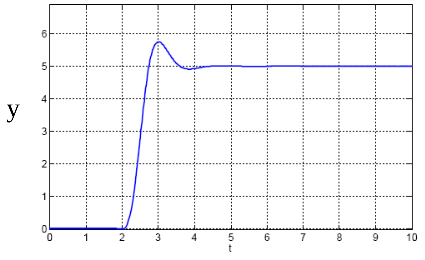
If we choose the eigenvalues -10, -8, -6, -5 for the observer dynamics, we get:

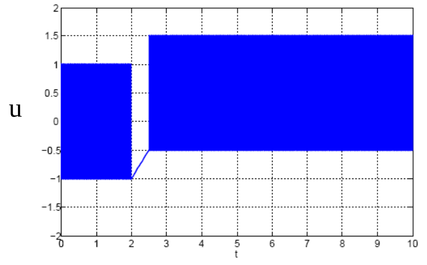
$$L = [-0.0025\ \ -0.0250\ \ 0.0175\ \ 0.2550]'$$

and $0<\lambda_0<5$

STATE NOT AVAILABLE: A NUMERICAL EXAMPLE

$$y^\circ(t) = 5sca(t-2), q = 1, r = 0\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$$
$$\hat{x}(0) = x(0) = [0\ 0\ 0\ 0]', \hat{y}(0) = y(0) = 0$$





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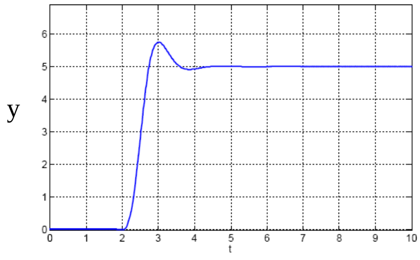
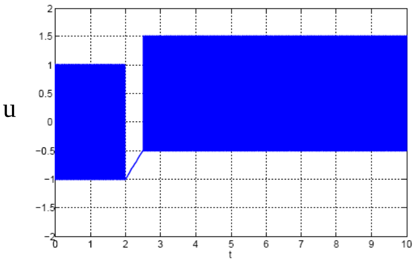
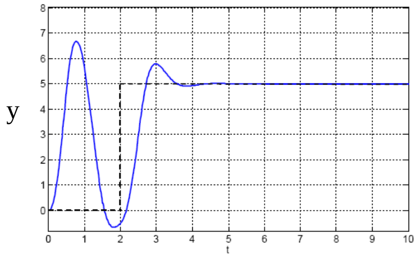


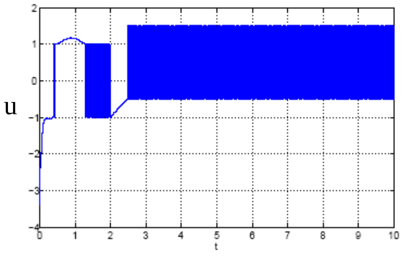


Same behavior as without the observer since the estimation error is zero at time 0 and then keeps being zero.

STATE NOT AVAILABLE: A NUMERICAL EXAMPLE

$$y°(t) = 5sca(t-2), q = 1, r = 0 \ [B_{MB/2} = 0.02, M_{MB/2} = 1]$$
$$x(0) = [0\ 0\ 0\ 0]', y(0) = 0; \ \ \hat{x}(0) = \ [0.012\ 0\ 0\ 0]', \hat{y}(0) = -24$$





STATE NOT AVAILABLE: A NUMERICAL EXAMPLE

$$y^{\circ}(t) = 5sca(t-2), q = 1, r = 0 \; [B_{MB/2} = 0.02, M_{MB/2} = 1]$$
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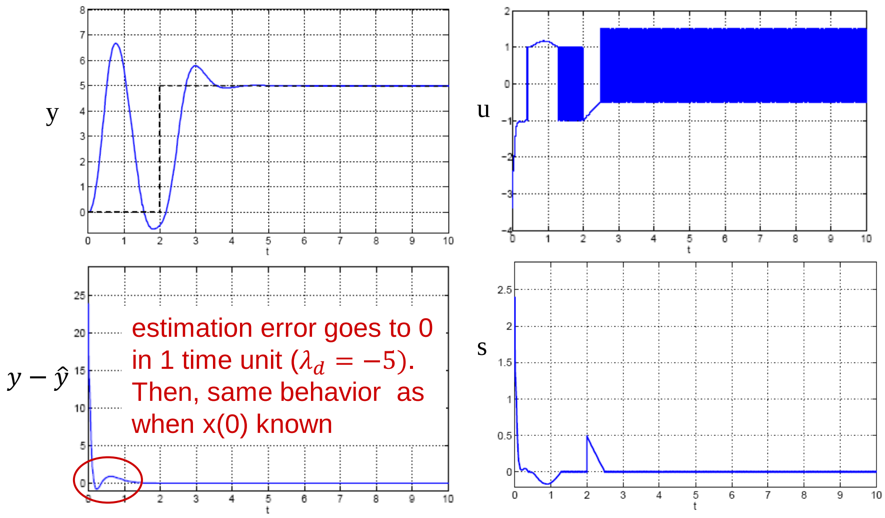

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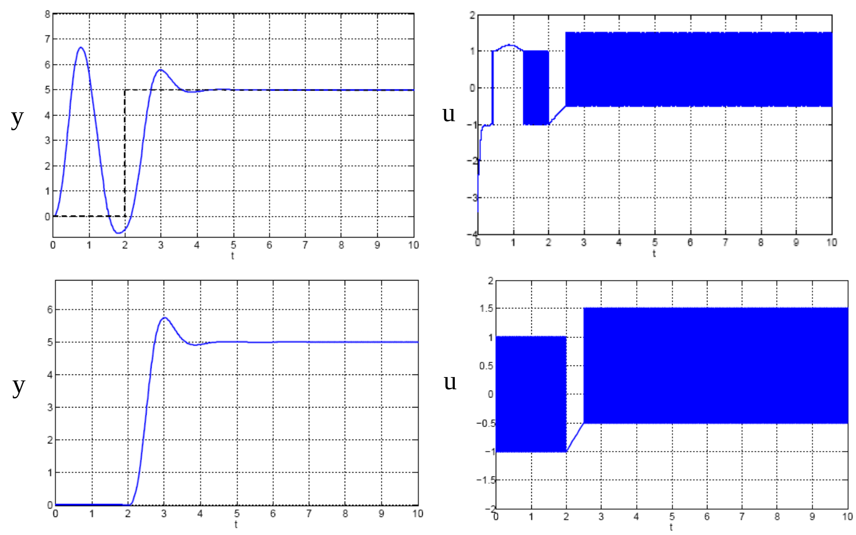
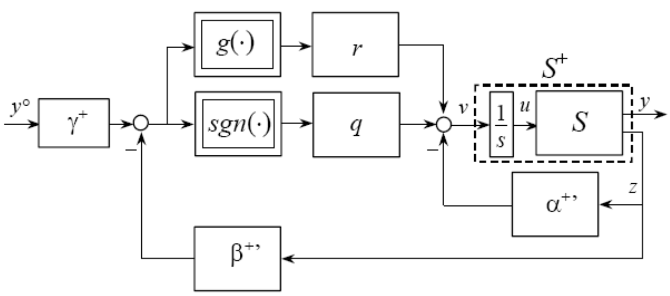


ISSUE OF THE HIGH FREQUENCY INPUT SWITCHING

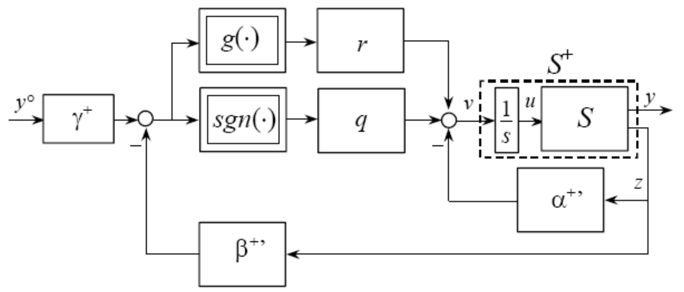
The high frequency switching of the control input in the sliding mode phase can be undesirable and even unacceptable.

ISSUE OF THE HIGH FREQUENCY INPUT SWITCHING



Possible solution:
filtering the high frequency components of the control input by introducing an auxiliary control variable v whose integral is the actual control variable u

A NUMERICAL EXAMPLE



S^+ controllable and observable system (no pole-zero cancellations)

$$\mathcal{G}(s) = \frac{-400\,s - 2000}{s^5 + 14\,s^4 + 55\,s^3 + 130\,s^2 - 200\,s} =$$

$$= \frac{10\,(1 + 0.2\,s)}{s\,(1 + 0.25\,s + 0.05\,s^2)(1 - s)(1 + 0.1\,s)}$$

A NUMERICAL EXAMPLE

$$S^+ : \begin{cases} \dot{u} = v \\ S : \begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x \end{cases} \end{cases}$$

enlarged controlled system

$z := [x' \quad u]' \qquad y^+ := [y \quad u]'$

$$S^+ : \begin{cases} \dot{z} = F\,z + G\,v \\ y^+ = H\,z \end{cases}$$

$$F := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \;,\quad G := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;,\quad H := \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix}$$

A NUMERICAL EXAMPLE

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$$F := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \quad , \quad G := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \quad H := \begin{bmatrix} \boxed{\begin{matrix} C & \quad 0 \end{matrix}} \\ \begin{matrix} 0 & \quad 1 \end{matrix} \end{bmatrix} H_1$$

$(F, \;\; G, \;\; H_1)$ not in controllable form

A NUMERICAL EXAMPLE

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 200 & -130 & -55 & -14 & 1 \end{bmatrix} \quad , \qquad [T = \tilde{M}_c M_c^{-1}]$$

$$\tilde{F} = T\,F\,T^{1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 200 & -130 & -55 & -14 \end{bmatrix} \quad , \qquad \tilde{G} = T\,G = G$$

$$\tilde{H} = H\,T^{1} = \begin{bmatrix} -2000 & -400 & 0 & 0 & 0 \\ -200 & 130 & 55 & 14 & 1 \end{bmatrix}$$

A NUMERICAL EXAMPLE

Dynamics within the sliding surface:

$$\chi^*(\lambda) = (\lambda^2 + 5\,\lambda + 20)(\lambda + 10)^2 = \lambda^4 + 25\,\lambda^3 + 220\,\lambda^2 + 900\,\lambda + 2000$$

$$\tilde{\beta}^{+\prime} := [\tilde{\beta}_4^+ \quad \tilde{\beta}_3^+ \quad \tilde{\beta}_2^+ \quad \tilde{\beta}_1^+ \quad 1] = [2000 \quad 900 \quad 220 \quad 25 \quad 1]$$

$$\tilde{\alpha}^{+\prime} := \tilde{\beta}^{\prime}\,\tilde{F} = [0 \quad 2200 \quad 770 \quad 165 \quad 11]$$

$$\tilde{\beta}_4^+ = 2000, \quad \tilde{b}_5^+ = \tilde{H}(1,1) = -2000, \qquad \gamma^+ = \tilde{\beta}_4^+/\tilde{b}_5^+ = -1.$$

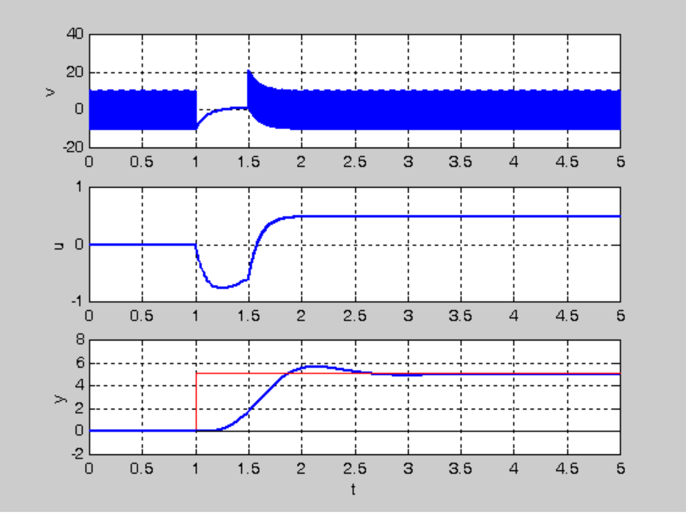
In the original state variables:

$$\beta^{+\prime} = \tilde{\beta}^{+\prime}\,T = [2200 \quad 770 \quad 165 \quad 11 \quad 1]$$

$$\alpha^{+\prime} = \tilde{\alpha}^{+\prime}\,T = [2200 \quad 770 \quad 165 \quad 11 \quad 11].$$

A NUMERICAL EXAMPLE

$$y°(t) = 5sca(t-1), \; q = 10 \; r = 0[B_{MB/2} = 0.02, M_{MB/2} = 1]$$



STABILIZATION OF DOUBLE INTEGRATOR

System S:

$$\dot{x}_1 = u$$
$$\dot{x}_2 = v$$
$$\dot{x}_3 = x_1 v - x_2 u$$

Goal: design a feedback controller that globally asymptotically stabilizes the origin

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Goal: design a feedback controller that globally asymptotically stabilizes the origin

Difficult to stabilize even locally. Indeed, linearization in the vicinity of the origin gives a non-controllable system:

$$\dot{x}_1 = u$$
$$\dot{x}_2 = v$$
$$\dot{x}_3 = 0$$

If x_1 and x_2 are too close to zero, x_3 cannot be steered to zero

STABILIZATION OF DOUBLE INTEGRATOR

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$$\dot{x}_3 = x_1 v - x_2 u$$

Goal: design a feedback controller that globally asymptotically stabilizes the origin

We adopt a sliding mode approach

STABILIZATION OF DOUBLE INTEGRATOR

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$$\dot{x}_1 = u$$
$$\dot{x}_2 = v$$
$$\dot{x}_3 = x_1 v - x_2 u$$

Idea: make x_3 converge faster to zero than x_1 and x_2

Sliding mode control:

$$u = -x_1 + x_2 sgn(x_3)$$
$$v = -x_2 - x_1 sgn(x_3)$$

[sliding surface x_3=0]

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[sliding surface x_3=0]

We first show that there exists a set of initial conditions such that trajectories starting there converge to the origin.

STABILIZATION OF DOUBLE INTEGRATOR

System S:

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Sliding mode control:

$$u = -x_1 + x_2 sgn(x_3)$$
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Control system:

$$\dot{x}_1 = -x_1 + x_2 sgn(x_3)$$
$$\dot{x}_2 = -x_2 - x_1 sgn(x_3)$$
$$\dot{x}_3 = -({x_1}^2 + {x_1}^2) sgn(x_3)$$

STABILIZATION OF DOUBLE INTEGRATOR

Control system:

$$\dot{x}_1 = -x_1 + x_2 sgn(x_3)$$
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Lyapunov function for the (x_1, x_2) space

$$V(x_1, x_2) = \frac{1}{2}({x_1}^2 + {x_2}^2)$$
$$\frac{dV}{dt} = x_1\dot{x}_1 + x_2\dot{x}_2 = -({x_1}^2 + {x_2}^2) = -2\mathrm{V}$$
$$V(t) = V(0)e^{-2t}, t \geq 0$$

→ x_1, x_2 tend to zero asymptotically

STABILIZATION OF DOUBLE INTEGRATOR

Control system:

$$\dot{x}_1 = -x_1 + x_2 sgn(x_3)$$
$$\dot{x}_2 = -x_2 - x_1 sgn(x_3)$$
$$\dot{x}_3 = -({x_1}^2 + {x_1}^2) sgn(x_3)$$

Lyapunov function for the (x_1, x_2) space

$$V(x_1, x_2) = \frac{1}{2}({x_1}^2 + {x_2}^2)$$

→ x_1, x_2 tend to zero asymptotically

As for x_3:

$$\dot{x}_3 = -({x_1}^2 + {x_1}^2) sgn(x_3) = -2V sgn(x_3)$$

STABILIZATION OF DOUBLE INTEGRATOR

As for x_3:

$$\dot{x}_3 = -({x_1}^2 + {x_1}^2)sgn(x_3) = -2Vsgn(x_3)$$

$$\frac{d|x_3|}{dt} = \dot{x}_3 sgn(x_3) = -2V$$

$$|x_3(t)| - |x_3(0)| = -2\int_0^t V(\tau)d\tau$$

$$|x_3(t)| = |x_3(0)| - 2\int_0^t V(\tau)d\tau$$

→ If $|x_3(0)| < 2\int_0^\infty V(\tau)d\tau$, then x_3 tends to zero in finite time

If $|x_3(0)| = 2\int_0^\infty V(\tau)d\tau$, then x_3 tends to zero in infinite time

STABILIZATION OF DOUBLE INTEGRATOR

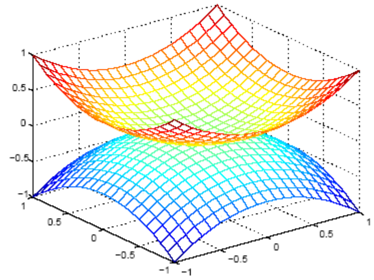
$$V(t) = V(0)e^{-2t}, t \geq 0$$

Then, $2\int_0^\infty V(\tau)d\tau = V(0) = \frac{1}{2}({x_1}^2(0) + {x_2}^2(0))$

Hence: $|x_3(0)| \leq 2\int_0^\infty V(\tau)d\tau$

becomes

$$|x_3(0)| \leq \frac{1}{2}({x_1}^2(0) + {x_2}^2(0))$$



STABILIZATION OF DOUBLE INTEGRATOR

If initial state satisfies

$$|x_3(0)| \leq \frac{1}{2}({x_1}^2(0) + {x_2}^2(0)) \qquad (*)$$

then the sliding mode controller leads the state to the origin

If the initial state doesn't satisfy that condition, then, apply constant control to drive it in that region:

$$\dot{x}_1 = \bar{u}$$
$$\dot{x}_2 = \bar{v}$$
$$\dot{x}_3 = x_1\bar{v} - x_2\bar{u}$$

$$x_1(t) = \bar{u}t + x_1(0)$$
$$x_2(t) = \bar{v}t + x_2(0)$$
$$x_3(t) = t(\bar{v}x_1(0) - x_2(0)\bar{u}) + x_3(0)$$

Then equation $(*)$ will be satisfied at some finite time instant t.

STABILIZATION OF DOUBLE INTEGRATOR

If initial state satisfies

$$|x_3(0)| \leq \frac{1}{2}({x_1}^2(0) + {x_2}^2(0))$$

then the sliding mode controller leads the state to the origin

If the initial state satisfies

$$|x_3(0)| > \frac{1}{2}({x_1}^2(0) + {x_2}^2(0))$$

then, apply constant control to drive it in the complementary region

→ Hybrid control scheme