Variable Structure Control (VSC)

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Lectures 1 and 2

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VARIABLE STRUCTURE CONTROL

Control strategy where

- a discontinuous feedback control law is designed that forces the state of the system to reach and then remain on a certain surface (the sliding surface);
- the dynamic of the system restricted to the sliding surface should produce a desired behavior, e.g., convergence to some suitable equilibrium [stable sliding mode].

It is also known as sliding mode control.

EXAMPLE [S.V. Emelianov]

S: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + 2x_2 + u \\ y = x_1 \end{cases}$ linear system

C: $u = -\psi(x)y$ switching controller

where $\psi(x) = \begin{cases} -4, & s(x) < 0 \\ +4, & s(x) > 0 \end{cases}$, $s(x) = x_1(0.5x_1 + x_2)$

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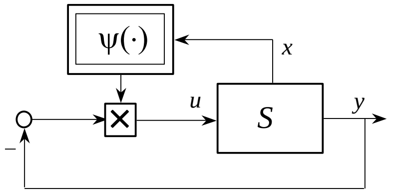
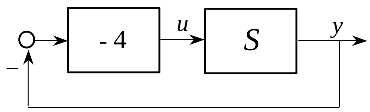


Σ: control system

EXAMPLE

- System Σ has a variable structure

If $s(x) < 0$



$\Sigma_1 : \psi(x) = - 4$

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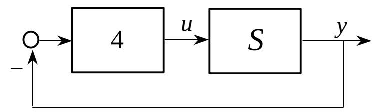

$\Sigma_2 : \psi(x) = 4$

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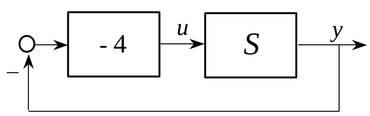

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$$\Sigma_1 \colon \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 3x_1 + 2x_2 \end{cases}$$

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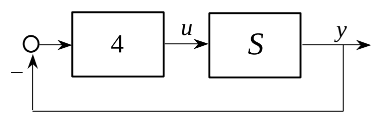

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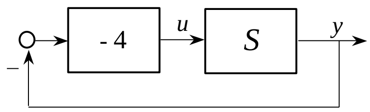

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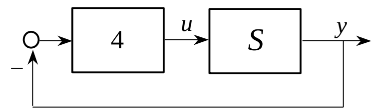

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$$\Sigma_1\colon \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 3x_1 + 2x_2 \end{cases}$$

$$\chi_1(\lambda) = \lambda^2 - 2\lambda - 3$$

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$$\chi_2(\lambda) = \lambda^2 - 2\lambda + 5$$

- Both systems are unstable

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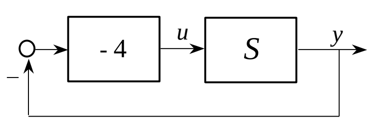

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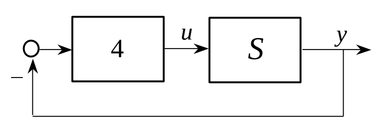

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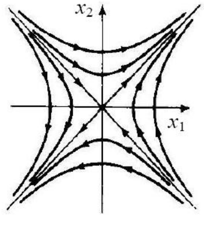
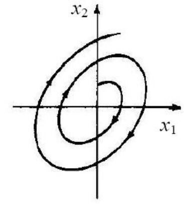


Σ_1

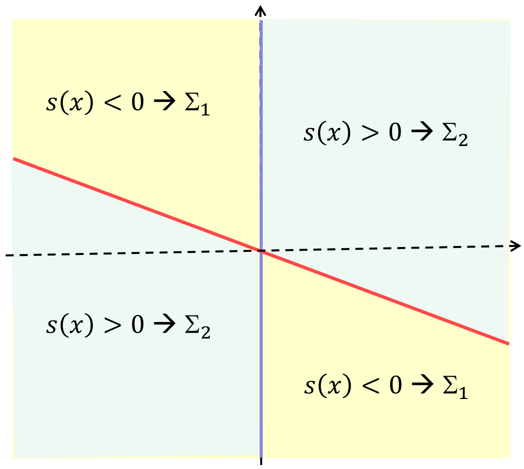


Σ_2

EXAMPLE

$$s(x) := x_1\,(0.5\,x_1 + x_2) = 0 \quad \Leftrightarrow \quad \begin{cases} \boxed{x_1 = 0} \\ \boxed{0.5\,x_1 + x_2 = 0} \end{cases}$$

switching surfaces



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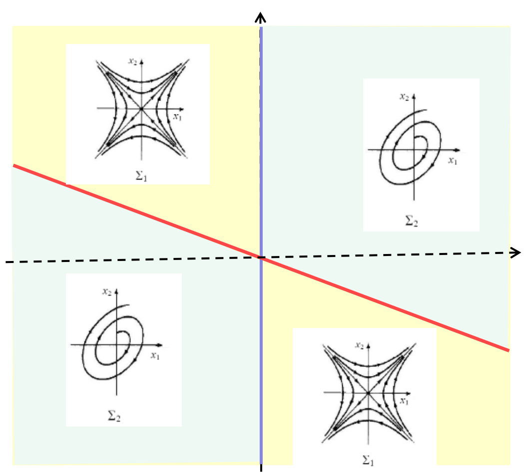

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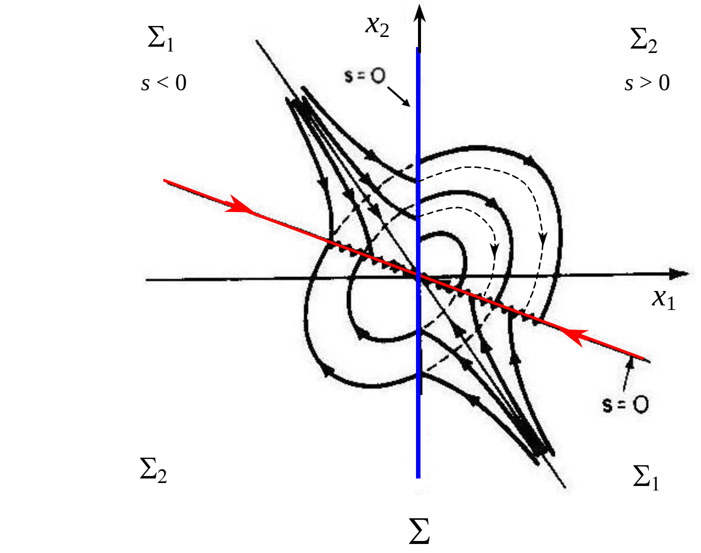

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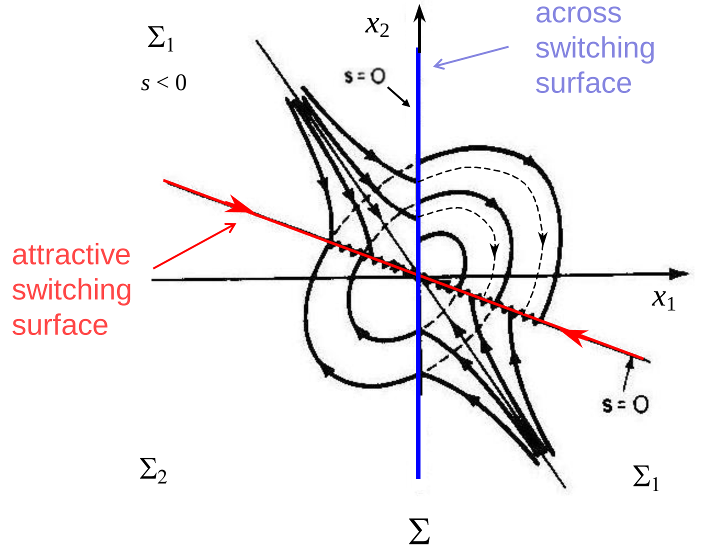

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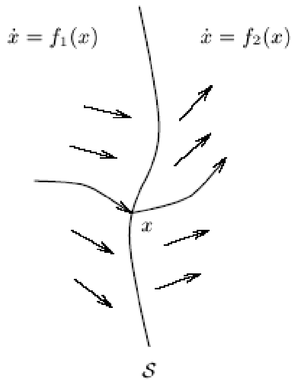
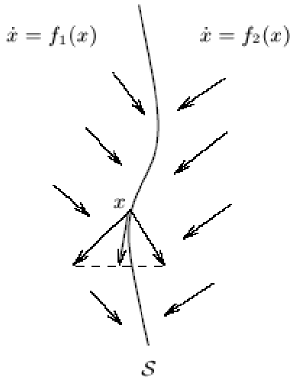


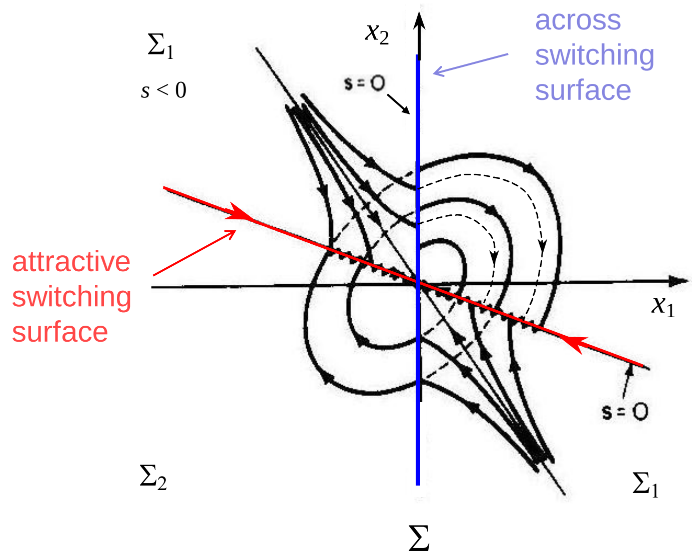
Across switching surface:
the state reaches the surface *S* while following some dynamics, crosses it, and continues its evolution according to the other dynamics

EXAMPLE



Attractive switching surface:
the state reaches the surface *S* and cannot leave it because the vector fields on both sides are pointing towards *S*
→ It can only slide along *S* (sliding mode)

EXAMPLE



EXAMPLE

After reaching the surface

$$0.5\,x_1 + x_2 = 0$$

infinitely fast switching occurs (ideal sliding mode) and the state is constrained to evolve on that surface.

$$\text{S:}\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -x_1 + 2x_2 + u\\ y = x_1\end{cases} \quad \& \quad x_2 = -0.5x_1$$

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$$\dot{x}_1 = -0.5\,x_1$$

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as. stable reduced order system

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The origin x = 0 is a globally asymptotically stable (pseudo-) equilibrium for Σ

VARIABLE STRUCTURE CONTROL: THE BASICS

Given a linear time-invariant SISO system S of order n

S: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

with (A,B) controllable and (A,C) observable, design a variable structure controller such that $y(t)$ tends to some (constant) reference signal $y°$ in some reasonable amount of time, for all $y°$ and for all $x(0)$.

VARIABLE STRUCTURE CONTROL: THE BASICS

Suppose that S is in the controllable canonical form:

$$S: \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) \end{cases},$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \dots & b_2 & b_1 \end{bmatrix}$$

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Suppose that S is in the controllable canonical form:

$$\dot{x}_i = x_{i+1} \quad , \qquad i = 1, 2, \dots , n\text{-}1$$

$$\dot{x}_n = - a_n x_1 - a_{n\text{-}1} x_2 - \dots - a_1 x_n + u$$

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Then, its transfer function is given by

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^{n\text{-}1} + b_2 s^{n\text{-}2} + \dots + b_n}{s^n + a_1 s^{n\text{-}1} + a_2 s^{n\text{-}2} + \dots + a_n}$$

with $b(s)$ and $a(s)$ coprime since (A,C) is observable.

For the output regulation problem to be well-posed $b_n \neq 0$ since otherwise we shall have $s = 0$ as a zero for $\mathrm{G}(s)$.

VARIABLE STRUCTURE CONTROL: THE BASICS

Design procedure:

1. Determine a switching function

$$s(\cdot) \;:\; \mathbf{R}^n \to \mathbf{R}$$

   such that S constrained on the sliding surface $s(x) = 0$ converges to a (pseudo-)equilibrium with $y = y°$.

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CHOICE OF THE SWITCHING FUNCTION

Typical choice (not only for linear systems) for the switching function:

$$s(x) := \beta_{n-1} x_1 + \beta_{n-2} x_2 + \ldots + \beta_1 x_{n-1} + x_n - \bar{w},$$

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Then, the system dynamics on the sliding surface $s(x) = 0$ is given by:

$$S^* : \quad \begin{cases} \dot{x}_i = x_{i+1} \quad , \qquad i = 1, 2, \ldots, n\text{-}2 \\ \dot{x}_{n-1} = x_n = -\beta_{n-1} x_1 - \beta_{n-2} x_2 - \ldots - \beta_1 x_{n-1} + \bar{w}\,. \end{cases}$$

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This is an (n-1)-dimensional system in controllable canonical form, whose characteristic polynomial is

$$\chi^*(\lambda) = \lambda^{n-1} + \beta_1 \lambda^{n-2} + \ldots + \beta_{n-1}$$

CHOICE OF THE SWITCHING FUNCTION

The roots of

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If all roots have strictly negative real part (thus, $\beta_{n-1} \neq 0$), then, S* is asymptotically stable and admits a single equilibrium for each value for $\overline{w}$

CHOICE OF THE SWITCHING FUNCTION

$$\overline{x}_1 = \overline{w}/\beta_{n-1} \; ; \qquad \overline{x}_i = 0 \;\; , \quad i = 2, 3, \dots, n\text{-}1$$

is the only asymptotically stable equilibrium of

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Correspondingly,

$$\bar{x}_n = 0 \,, \qquad \bar{y} = C\,\bar{x} = b_n\,\bar{x}_1 = \bar{w}\; b_n/\beta_{n\text{-}1}$$

If we set $\bar{w} = \gamma\, y°, \gamma := \beta_{n\text{-}1}/b_n$, then $\bar{y} = y°$.

VARIABLE STRUCTURE CONTROL: THE BASICS

Design procedure:

1. Determine a switching function

   $$s(\cdot) \;:\; \mathbf{R}^n \to \mathbf{R}$$

   such that S constrained on the sliding surface $s(x) = 0$ converges to a (pseudo-)equilibrium with $\mathrm{y} = y°$.

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Here we shall see one possible solution to the problem.

REACHING CONDITION

We shall adopt the so-called “reaching-law approach” to impose the reaching condition.

REACHING-LAW APPROACH

Specify the dynamics of the switching function $s(x(t))$ so that the Lyapunov-like function

$$V(s) = \frac{1}{2}s^2,$$

has negative time derivative satisfying

$$\frac{dV}{dt} = s\dot{s} \leq -\eta\,|s|, \qquad \eta > 0$$

Statement:
For any initial condition $x(0)$, $s(x(t))$ converges to zero in finite time.

REACHING-LAW APPROACH

Proof [finite time convergence]

Given that $V(s) = \frac{1}{2}s^2$,

$$\frac{dV}{dt} = s\dot{s} \leq -\,\eta\,|s| = -\eta\sqrt{2V}$$

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and, hence,

$$\int_0^t \frac{1}{2}\frac{\dot{V}}{\sqrt{V}}\,dt \leq \int_0^t -\frac{\eta}{\sqrt{2}}dt \;\; \rightarrow \sqrt{V(t)} - \sqrt{V(0)} \leq -\frac{\eta}{\sqrt{2}}\mathrm{t}$$

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This can be rewritten as

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$$\frac{1}{\sqrt{2}}|s(x(t))| \leq -\frac{\eta}{\sqrt{2}}\mathrm{t} + \frac{1}{\sqrt{2}}|s(x(0))|$$

and the time required to reach $\mathrm{s} = 0$ is upper bounded by

$$\frac{|s(x(0))|}{\eta}$$

REACHING-LAW APPROACH

Dynamics of the switching function:

$$\dot{s} = -q\ sgn(s) - rg(s)$$

with $q > 0$ and $r \geq 0$, and $g(.)$ such that $\boxed{sg(s) > 0}$, $\forall s \neq 0$.

$$\begin{cases} -rg(s) > 0, & s < 0 \\ -rg(s) < 0, & s > 0 \end{cases}$$

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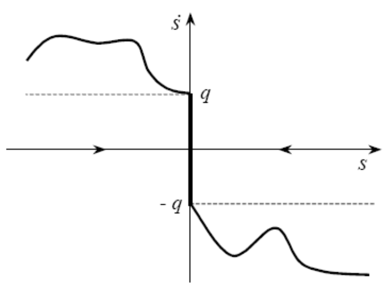

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The condition for finite time convergence to the switching surface is satisfied with $\eta = q$:

$$\frac{dV}{dt} = s\dot{s} = -q\, sgn(s)s - rsg(s) = -q|s| - rsg(s) \leq -q|s|$$

The time to convergence satisfies

$$t_r \leq \frac{|s(x(0))|}{q}$$

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What is the control law that imposes this dynamics to s(t)?

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$s(x) := \beta_{n\text{-}1}\, x_1 + \beta_{n\text{-}2}\, x_2 + \ldots + \beta_1\, x_{n\text{-}1} + x_n - \bar{w}, \quad \bar{w} = \gamma\, y°, \quad \gamma := \beta_{n\text{-}1}/b_n,$

which can be rewritten in compact form as:

$s(x) = \beta'\, x - \gamma\, y° \quad , \quad \beta' := [\beta_{n\text{-}1} \quad \beta_{n\text{-}2} \quad \ldots \quad \beta_1 \quad 1]\,,$

$\dot{s} = \beta'\, \dot{x} = \beta'\, (A\, x + B\, u)\,.$

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$s(x) := \beta_{n\text{-}1}\, x_1 + \beta_{n\text{-}2}\, x_2 + \ldots + \beta_1\, x_{n\text{-}1} + x_n - \bar{w}, \quad \bar{w} = \gamma\, y°, \quad \gamma := \beta_{n\text{-}1}/b_n,$

which can be rewritten in compact form as:

$s(x) = \beta'\, x - \gamma\, y° \quad , \quad \beta' := [\beta_{n\text{-}1} \quad \beta_{n\text{-}2} \quad \ldots \quad \beta_1 \quad 1]\,,$

$\dot{s} = \beta'\, \dot{x} = \beta'\, (A\, x + B\, u)\,.$

$$\boxed{\beta'\, (A\, x + B\, u) = -\, q\ sgn(s(x)) - r\ g(s(x))}$$

REACHING-LAW APPROACH

$$\boxed{\beta' (A\,x + B\,u) = -\,q\,sgn(s(x)) - r\,g(s(x))}$$

$$\beta' := [\beta_{n-1} \quad \beta_{n-2} \quad \dots \quad \beta_1 \quad 1] \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix} \longrightarrow \beta' B = 1$$

REACHING-LAW APPROACH

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$$u = -\left(\beta' A\,x + q\,sgn(\beta' x - \gamma\,y^\circ) + r\,g(\beta' x - \gamma\,y^\circ)\right)$$

REACHING-LAW APPROACH

$$\boxed{\beta' (A\,x + B\,u) = -\,q\,sgn(s(x)) - r\,g(s(x))}$$

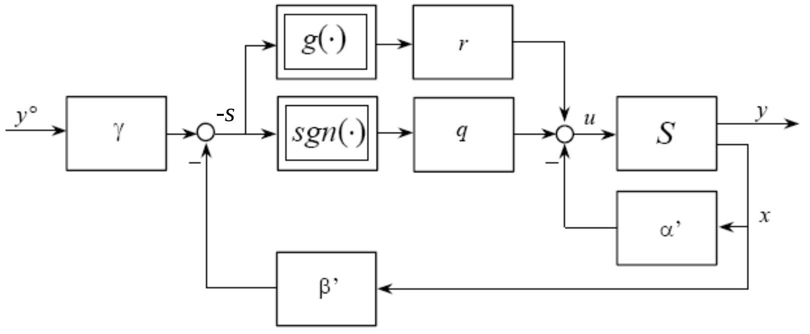
$$\beta' := [\beta_{n-1} \quad \beta_{n-2} \quad \dots \quad \beta_1 \quad 1] \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix} \longrightarrow \beta' B = 1$$

$$u = -\left(\beta' A\,x + q\,sgn(\beta' x - \gamma\,y^\circ) + r\,g(\beta' x - \gamma\,y^\circ)\right) =$$

$$= -\,\alpha' x + q\,sgn\,(\gamma\,y^\circ - \beta' x) + r\,g(\gamma\,y^\circ - \beta' x) \;.$$

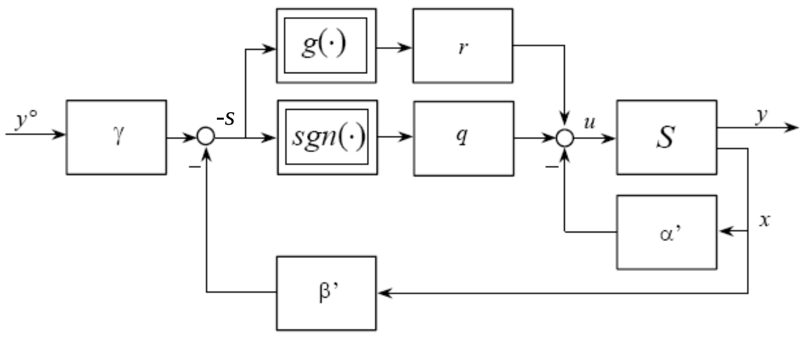
$\uparrow$

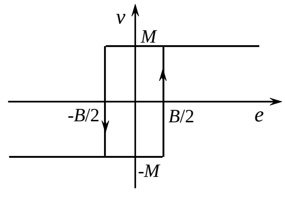
$g(-s) = -g(s);\ \alpha = \beta' A$



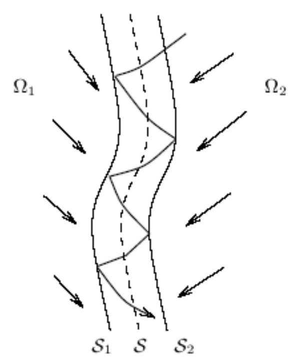
$$u = -\left(\beta' A\,x + q\,sgn(\beta' x - \gamma\,y^\circ) + r\,g(\beta' x - \gamma\,y^\circ)\right) =$$

$$= -\,\alpha' x + q\,sgn\,(\gamma\,y^\circ - \beta' x) + r\,g(\gamma\,y^\circ - \beta' x) \;.$$

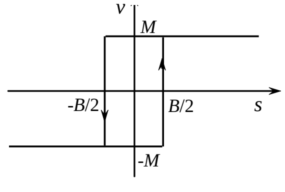




sgn function implemented as
an hysteresis switching controller
with M=1 and B $\rightarrow$ 0



infinitely fast switching
is avoided



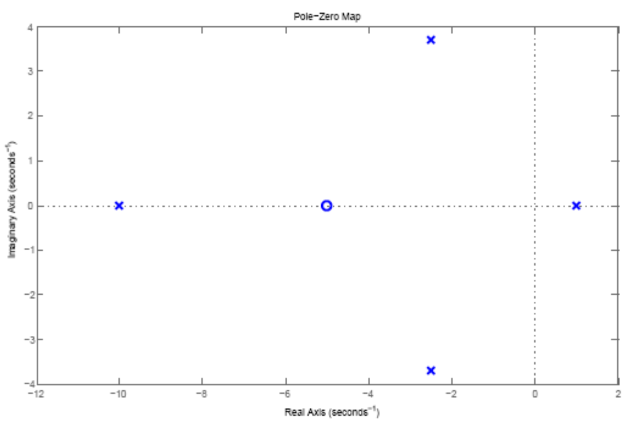
sgn function implemented as
an hysteresis switching controller
with M=1 and B $\rightarrow$ 0

A NUMERICAL EXAMPLE

$$G(s) = \frac{-400(s+5)}{(s^2+5s+20)(s+10)(s-1)}$$
$$= \frac{-400(s+5)}{s^4+14s^3+55s^2+130s-200}$$

A NUMERICAL EXAMPLE

$$G(s) = \frac{-400(s+5)}{(s^2+5s+20)(s+10)(s-1)}$$
$$= \frac{-400(s+5)}{s^4+14s^3+55s^2+130s-200}$$



A NUMERICAL EXAMPLE

$$G(s) = \frac{-400(s+5)}{s^4 + 14s^3 + 55s^2 + 130s - 200}$$

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 200 & -130 & -55 & -14 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = -[2000 \quad 400 \quad 0 \quad 0\ ]$$

$$a_1 = 14, \quad a_2 = 55, \quad a_3 = 130, \quad a_4 = -200$$

$$b_1 = b_2 = 0, \quad b_3 = -400, \quad b_4 = -2000.$$

A NUMERICAL EXAMPLE

Recall that the β coefficients of the switching function [n=4]

$$s(x) := \beta_{n-1}\,x_1 + \beta_{n-2}\,x_2 + \ldots + \beta_1\,x_{n-1} + x_n - \overline{w},$$

appear in the characteristic polynomial of the linear system S* of order 3 governing the dynamics of S when restricted to the sliding surface $s(x) = 0$:

$$\chi^*(\lambda) = \lambda^{n-1} + \beta_1\,\lambda^{n-2} + \ldots + \beta_{n-1}$$

A NUMERICAL EXAMPLE

Recall that the β coefficients of the switching function [n=4]

$$s(x) := \beta_{n-1} x_1 + \beta_{n-2} x_2 + \ldots + \beta_1 x_{n-1} + x_n - \bar{w}, \quad \bar{w} = \gamma y^\circ$$

appear in the characteristic polynomial of the linear system S* of order 3 governing the dynamics of S when restricted to the sliding surface $s(x) = 0$:

$$\chi^*(\lambda) = \lambda^{n-1} + \beta_1 \lambda^{n-2} + \ldots + \beta_{n-1}$$

If we set the eigenvalues of S* equal to those stable of S

$$\chi^*(\lambda) = (\lambda^2 + 5\lambda + 20)(\lambda + 10) = \lambda^3 + 15\lambda^2 + 70\lambda + 200$$

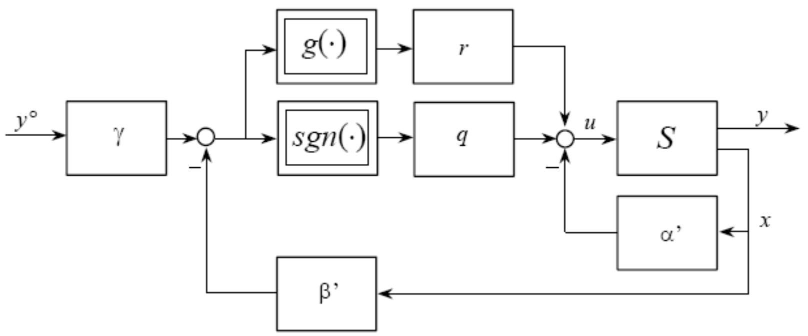
then,

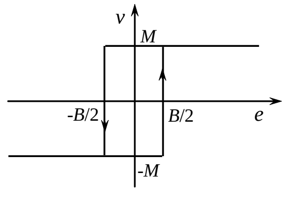
$$\beta' := [\beta_3 \quad \beta_2 \quad \beta_1 \quad 1] = [200 \quad 70 \quad 15 \quad 1]$$

So that $\quad \gamma = \beta_3 / b_4 = -0.1 \qquad \alpha' := \beta' A = [200 \quad 70 \quad 15 \quad 1]$

A NUMERICAL EXAMPLE

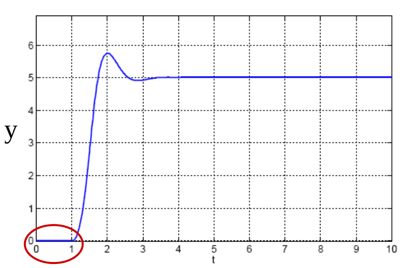
$$q = 1, r = 0$$

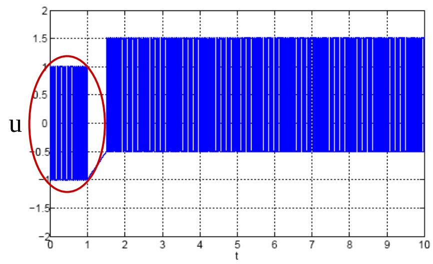




A NUMERICAL EXAMPLE

$y°(t) = 5sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

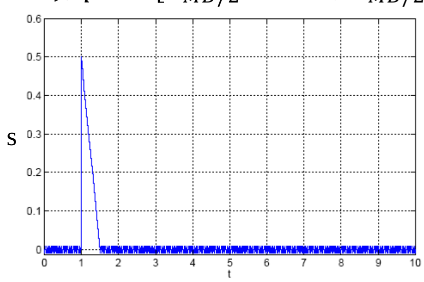




The system is initially on the sliding surface corresponding to $y° = 0$, at the (quasi)-equilibrium with $\bar{y} = y° = 0$, and keeps sliding on it in the time interval [0,1).
When $y° = 5$, we have a different sliding surface.
The time needed for reaching it satisfies $t_r \leq \frac{|s(x(1))|}{q}$

A NUMERICAL EXAMPLE

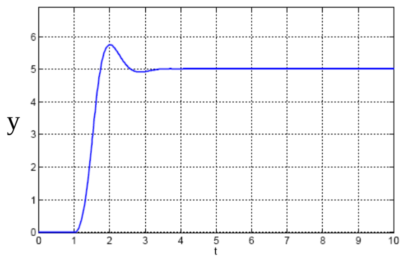
$y°(t) = 5sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

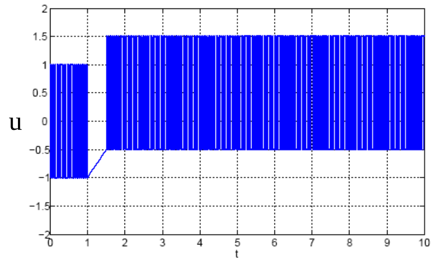


$$t_r \leq \frac{|s(x(1))|}{q} = 0.5$$

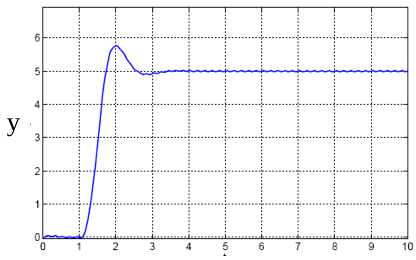
A NUMERICAL EXAMPLE

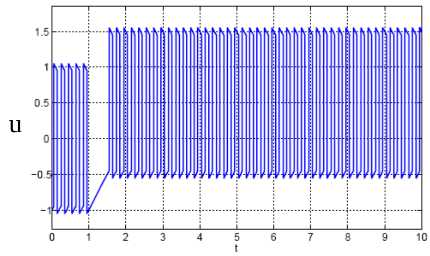
$y^{\circ}(t) = 5sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$





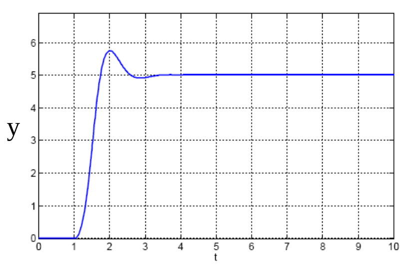
$y^{\circ}(t) = 5sca(t-1), q = 1\ [B_{MB/2} = 0.1, M_{MB/2} = 1]$

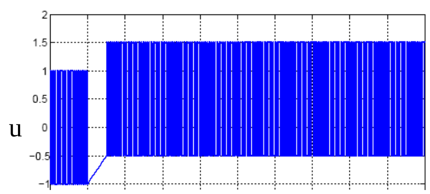




A NUMERICAL EXAMPLE

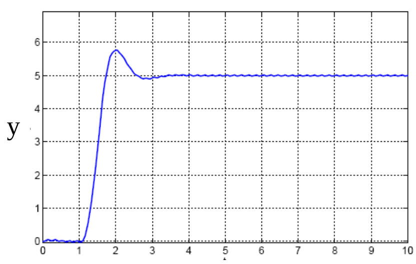
$y^{\circ}(t) = 5sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

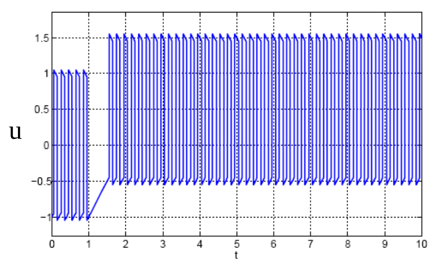




- slower switchings
- same duration of reaching phase

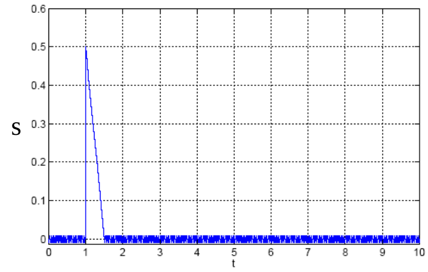
$y^{\circ}(t) = 5sca(t-1), q = 1\ [B_{MB/2} = 0.1, M_{MB/2} = 1]$



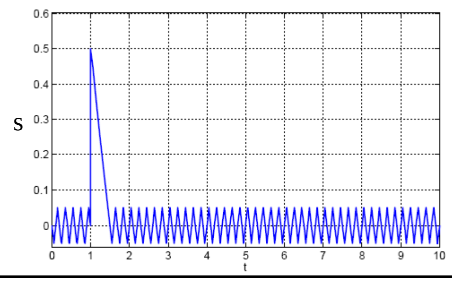


A NUMERICAL EXAMPLE

$y°(t) = 5sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

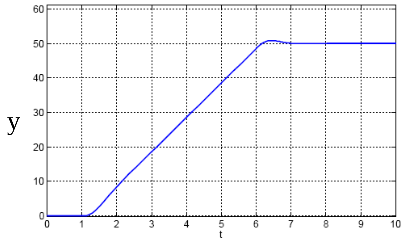


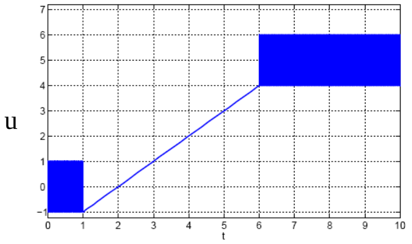
$y°(t) = 5sca(t-1), q = 1\ [B_{MB/2} = 0.1, M_{MB/2} = 1]$

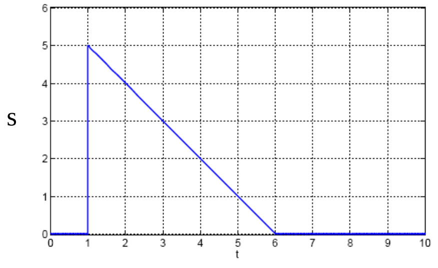


A NUMERICAL EXAMPLE

$y°(t) = 50sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$



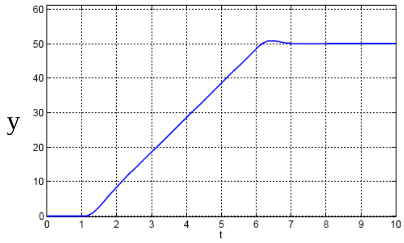


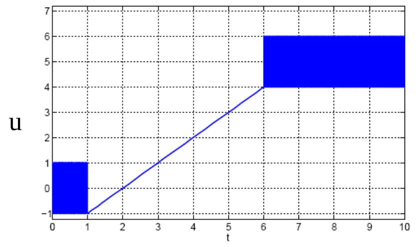


$$t_r \leq \frac{|s(x(0))|}{q} = 5$$

A NUMERICAL EXAMPLE

$y°(t) = 50sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

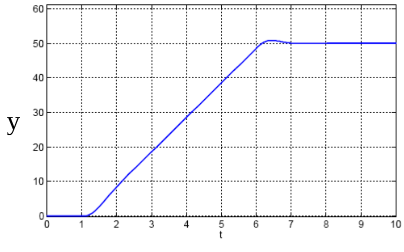


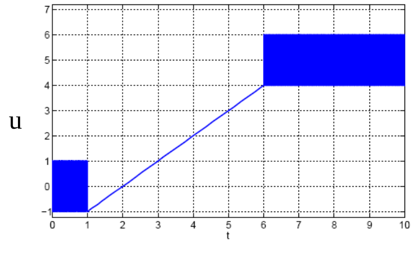


$y°(t) = 50sca(t-1), q = 3\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

A NUMERICAL EXAMPLE

$y°(t) = 50sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

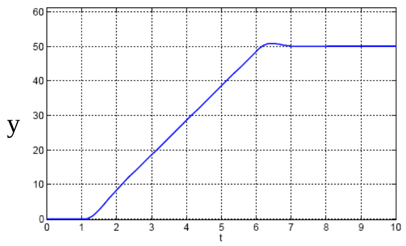


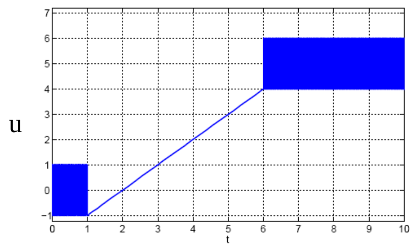


$y°(t) = 50sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 3]$

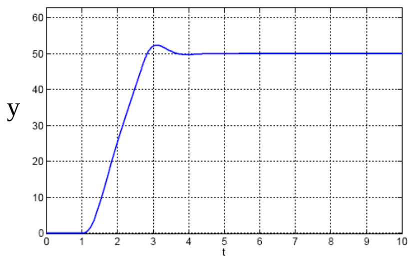
A NUMERICAL EXAMPLE

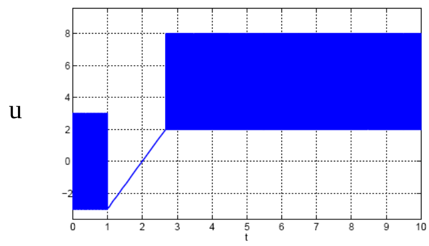
$y°(t) = 50sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$





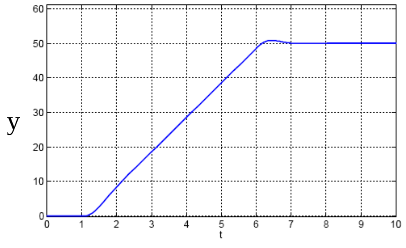
$y°(t) = 50sca(t-1), q = 3\ [B_{MB/2} = 0.1, M_{MB/2} = 1]$

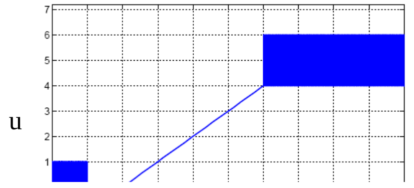




A NUMERICAL EXAMPLE

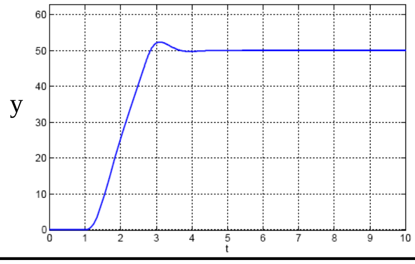
$y°(t) = 50sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

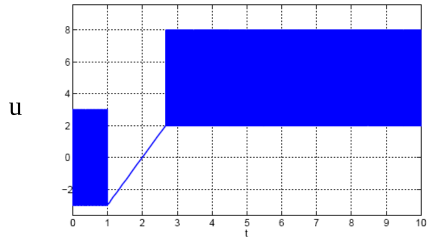




- smaller duration reaching phase
- but larger amplitude of u excursions

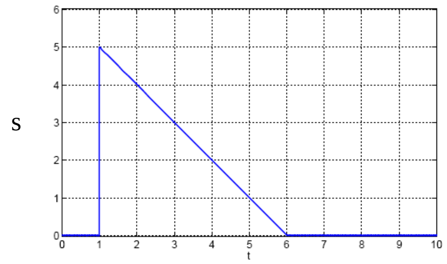
$y°(t) = 50sca(t-1), q = 3\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$



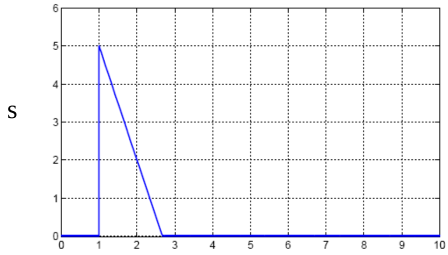


A NUMERICAL EXAMPLE

$y°(t) = 50sca(t-1), q = 1\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$



$y°(t) = 50sca(t-1), q = 3\ [B_{MB/2} = 0.02, M_{MB/2} = 1]$

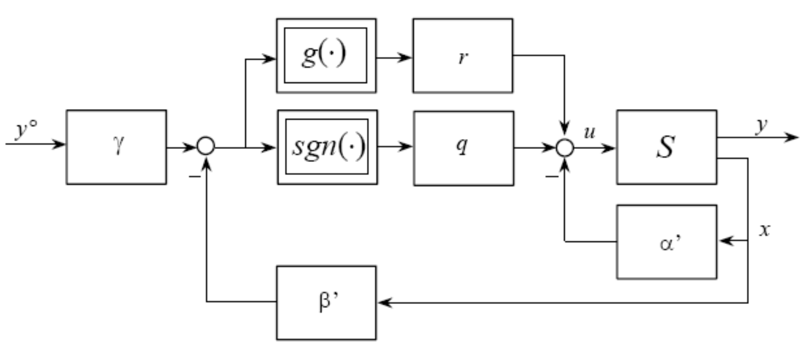


A NUMERICAL EXAMPLE

How can one reduce the duration of the reaching phase, while do not affecting the u excursion?

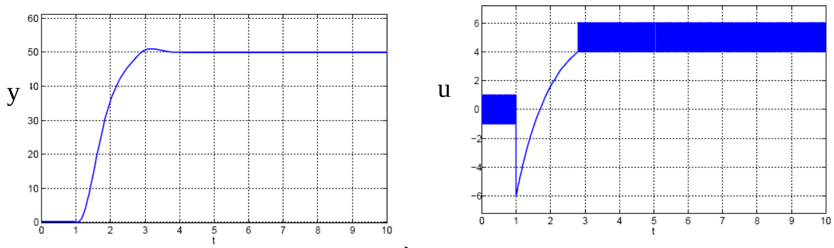
One can use an appropriate $g(.)$ function. Take, e.g.,

$$g(s) = s, \qquad s \in R; \qquad r = 1$$

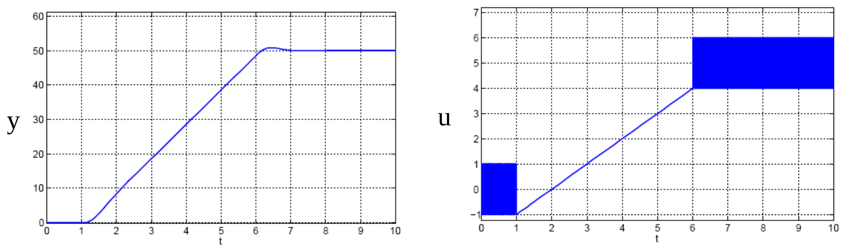


A NUMERICAL EXAMPLE

$$y^{\circ}(t) = 50sca(t-1), q = 1 \; [B_{MB/2} = 0.02, M_{MB/2} = 1],$$
$$g(s) = s, r = 1$$



$$g(s) = 0, r = 0$$



A NUMERICAL EXAMPLE

We shall now evaluate:

- impact of the choice of $\chi^*(\lambda)$ on the control law
- robustness of the control strategy with respect to parameter uncertainty

We shall fix $r = 0$, for simplicity.

A NUMERICAL EXAMPLE

Recall that the β coefficients of the switching function [n=4]

$$s(x) := \beta_{n-1} x_1 + \beta_{n-2} x_2 + \ldots + \beta_1 x_{n-1} + x_n - \bar{w}, \quad \bar{w} = \gamma y°$$

appear in the characteristic polynomial of the linear system S* of order 3 governing the dynamics of S when restricted to the sliding surface $s(x) = 0$:

$$\chi^*(\lambda) = \lambda^{n-1} + \beta_1 \lambda^{n-2} + \ldots + \beta_{n-1}$$

If we set the eigenvalues of S* all equal to -5

$$\chi^*(\lambda) = (\lambda + 5)^3 = \lambda^3 + 15\lambda^2 + 75\lambda + 125$$

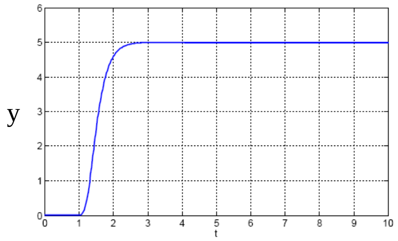
then,

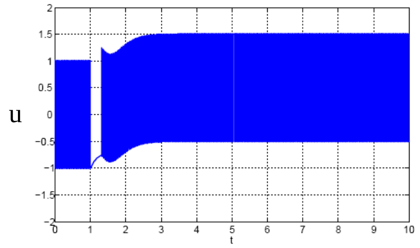
$$\beta' := [\beta_3 \quad \beta_2 \quad \beta_1 \quad 1] = [125 \quad 75 \quad 15 \quad 1]$$

So that $\alpha' := \beta' A = [200 \quad -5 \quad 20 \quad 1]$ $\quad \gamma = \beta_3/b_4 = -0.0625$

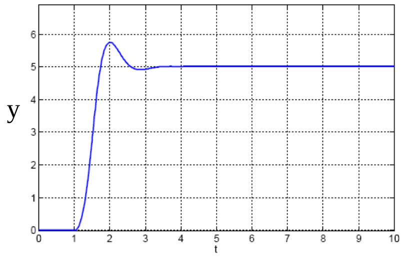
A NUMERICAL EXAMPLE

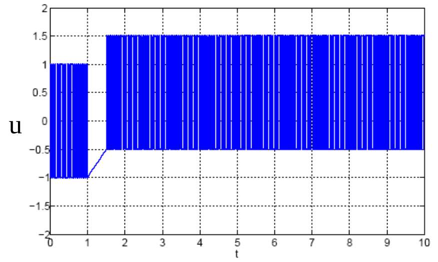
$y°(t) = 5sca(t-1), q = 1 \; [B_{MB/2} = 0.02, M_{MB/2} = 1] \quad \lambda^* = -5$





$y°(t) = 5sca(t-1), q = 1 \; [B_{MB/2} = 0.02, M_{MB/2} = 1]$

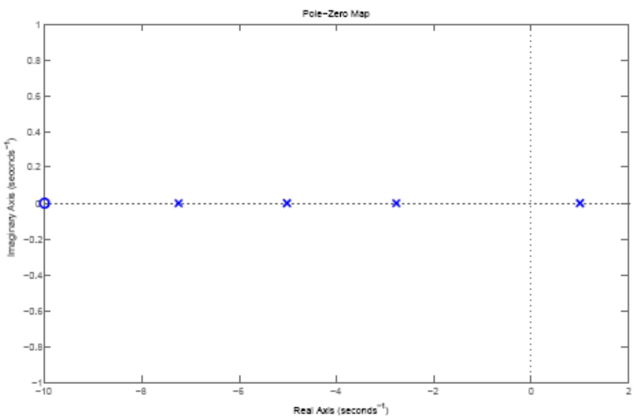




A NUMERICAL EXAMPLE

$$G(s) = \frac{-400(s+5)}{s^4 + 14s^3 + 55s^2 + 130s - 200}$$

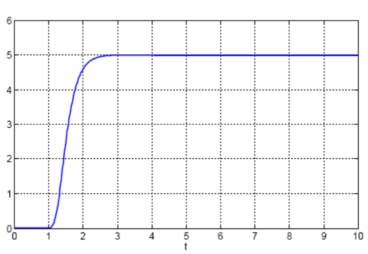
$$G_1(s) = \frac{-150(s+10)}{s^4 + 14s^3 + 55s^2 + 30s - 100}$$



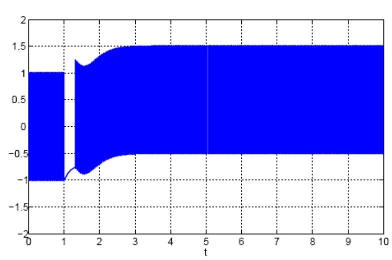
A NUMERICAL EXAMPLE

$y°(t) = 5sca(t-1), q = 1, \lambda^* = -5$, applied to $G(s)$

y

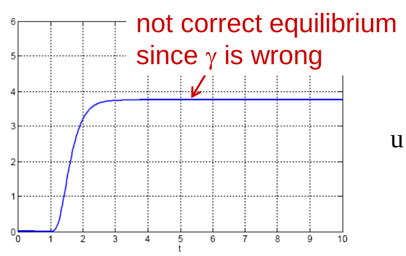


u

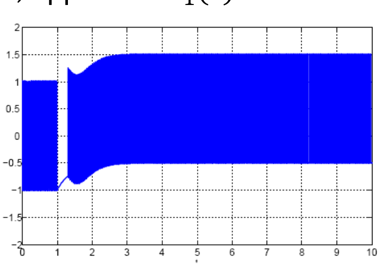


$y°(t) = 5sca(t-1), q = 1, \lambda^* = -5$, applied to $G_1(s)$

y



u



A NUMERICAL EXAMPLE

Scheme with adaptation of the gain γ: